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A note on a totally umbilical proper slant submanifold of a nearly Kaehler manifold

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ABSTRACT

In this paper, we study totally umbilical proper slant submanifolds of a nearly Kaehler manifold. We prove that every totally umbilical proper slant submanifold of a nearly Kaehler manifold is totally geodesic.

Keywords: Slant submanifold; totally-umbilical; totally geodesic; nearly Kaehler manifold.

INTRODUCTION

The notion of slant submanifolds of an almost Hermitian manifold was introduced by Chen (1990, 1991). These submanifolds are the generalization of both holomorphic and totally real submanifolds of an almost Hermitian manifold with an almost complex structure J. A nearly Kaehler structure on a manifold provides an interesting study with differential geometric point of view (Gray, 1969; Gray, 1970).

Consequently, the study of submanifolds of a nearly Kaehler manifold vis-à-vis that of a Kaehler manifold assumes significance in general, then the study of totally umbilical CR-submanifolds of a nearly Kaehler manifold has been studied in (Khan *et al.*, 1994). Later on, Khan & Khan (2007) extended this study to the semi-slant submanifold of a nearly Kaehler manifold. Recently, Sahin (2009) proved that every totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic. In this paper we prove that a totally umbilical proper slant submanifold of a nearly Kaehler manifold which has a nearly Kaehler structure is totally geodesic.

PRELIMINARIES

Let $\overline{M}$ be a Riemannian manifold with almost complex structure J and Hermitian metric g satisfying (Yano & Kon, 1984),

$$(a)\;\; J^2=-I, \quad \text{(b)}\;\; g(JX, JY)=g(X, Y), \tag{1}$$

for all vector fields $X, Y \in \Gamma(T\overline{M})$, where $\Gamma(T\overline{M})$ is the lie algebra of vector fileds on $\overline{M}$. If the almost complex structure J satisfies

$$\left(\overline{\nabla}_X J\right)X= 0 \tag{2}$$

for any $X, Y \in \Gamma(T\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$ then $\overline{M}$is said to have a *nearly Kaehler structure*. In this case $\overline{M}$ is *a nearly Kaehler* manifold. Equation (2) is equivalent to $\left(\overline{\nabla}_X J\right)Y + \left(\overline{\nabla}_Y J\right)X= 0$. Obviously, every Kaehler manifold is nearly Kaehler. The geometric meaning of nearly Kaehler condition is that geodesics are holomorphically planer curves. So far as non Kaehler, nearly Kaehler manifolds are concerned, one of the most prominent example is that of S^6 on a 6-dimensional unit sphere S^6 (Gray, 1969).

The covariant diffferentiation of the almost complex structure J is defined as

$$\left(\overline{\nabla}_X J\right)Y = \overline{\nabla}_X \mathrm{JY} - \mathrm{J}\overline{\nabla}_X \mathrm{Y}, \tag{3}$$

for all $X, Y \in \Gamma(T\overline{M})$.

Let M be a submanifold of a Riemannian manifold $\overline{M}$ and let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(T^{\perp}M)$ the set of all vector fields normal to M, then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y=\nabla_X Y+h(X, Y), \tag{4}$$

$$\overline{\nabla}_X N=-A_N X+\nabla^{\perp}_X N \tag{5}$$

for any $X, Y \in \Gamma(TM)$, where ∇ is the induced Riemannian connection on M, N is vector field normal to M, h is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and A_N is the shape operator of second fundamental form. Moreover, we have

$$g(A_{\mathrm{N}}X, Y) =g(h(X, Y),N). \tag{6}$$

where g denotes the Riemanian metric on $\overline{M}$ as well as the metric induced on M. The mean curvature vector H on M is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} \mathrm{h}(\mathrm{e_i}, \mathrm{e_i})$$

where n the dimension of M and $(e_1, e_2, , e_n)$ is a local orthonormal frame of vector fields on M.

A submanifold M of a Riemannian manifold $\overline{M}$ is said to be *totally umbilical* if

$$h(X, Y) = g(X, Y)H. \tag{7}$$

If $h(X, Y) = 0$ for any $X, Y \in \Gamma(TM)$, then the submanifold is said to be *totally geodesic*.

For any $X \in \Gamma(TM)$, we write

$$JX = TX + FX, \tag{8}$$

where TX and FX are tangential and normal components of JX, respectively. Similarly, for any vector field N normal to M, we put

$$JN = tN + fN, \tag{9}$$

where tN and fN are the tangential and normal components of JN, respectively.

The covariant derivatives of T, F, t and f are

$$\left(\overline{\nabla}_X T\right) Y = \nabla_X \mathrm{TY} - \mathrm{T}\nabla_X \mathrm{Y}, \tag{10}$$

$$\left(\overline{\nabla}_X F\right) Y = \nabla_X^{\perp} \mathrm{FY} - \mathrm{F}\nabla_X \mathrm{Y}, \tag{11}$$

$$\left(\overline{\nabla}_X t\right) N = \nabla_X \mathrm{tN} - \mathrm{t}\nabla_X^{\perp} \mathrm{N}, \tag{12}$$

$$\left(\overline{\nabla}_X f\right) N = \nabla_X^{\perp} \mathrm{fN} - \mathrm{f}\nabla_X^{\perp} \mathrm{N}, \tag{13}$$

for any $X, Y \in \Gamma(TM)$ and $\mathrm{N} \in \Gamma(T^{\perp}M)$.

Now, let us denote the tangential and normal parts of $\left(\overline{\nabla}_X J\right) Y$ by $P_X Y$ and $Q_X Y$, i.e.,

$$\left(\overline{\nabla}_X J\right) Y = P_X Y + Q_X Y$$

for any $X, Y \in \Gamma(TM)$. By and easy computation, we obtain the following formulae

$$P_X Y = \left(\overline{\nabla}_X T\right) Y - A_{FY}\mathrm{X}{-}\mathrm{th}(\mathrm{X},\mathrm{Y}), \tag{14}$$

$$Q_X Y = \left(\overline{\nabla}_X F\right) Y + h(X, tY) - -\mathrm{fh}(\mathrm{X},\mathrm{Y}). \tag{15}$$

Similarly, for any $\mathrm{N} \in \Gamma(T^{\perp}M)$, denote the tangential and normal parts of $\left(\overline{\nabla}_X J\right)N$ by $P_X N$ and $Q_X N$ respectively, we obtain

$$P_X N = \left(\overline{\nabla}_X t\right) N + T A_N \mathrm{X} {-} A_{fN}\mathrm{X}, \tag{16}$$

$$Q_X N = \left(\overline{\nabla}_X f\right) N + h(tN, X) + F A_N \mathrm{X}. \tag{17}$$

From now on, for any non-zero vector X tangent to M at x, the angle $\theta(X)$ between JX and $T_x M$ is called *Wirtinger angle* of X. It is easy to observe that *Wirtinger angle* $\theta(X)$ of X is in fact the angle between JX and TX. An immersion $f : M \to M$ is called *general slant imersion* if the angle $\theta(X)$ is constant (i.e., independent of the choice of $x \in M$ and $x \in T_x M$), in this case the constant Wirtinger angle is called slant angle. Holomorphic and totally real immersions with Wirtinger angle $\theta = 0$ and $\theta = \pi/2$. A general slant immersion which is neither holomorphic nor totally real is caled *proper slant immersion* with slant angle $\theta \in (0, \frac{\pi}{2})$.

A submanifold M of almost Hermition manifold $\overline{M}$ is slant submanifold of $\overline{M}$ if and only if (Chen, 1990)

$$T^2 = \lambda I \tag{18}$$

for some real number $\lambda \in [-1, 0]$, where I is the identity transformation of the tangent bundle TM of the submanifold M. Moreover, if M is a slant submanifold and θ is the slant angle of M, then $\lambda = -\cos^2\theta$. Hence, for a proper slant submanifold, we have

$$g(TX, TY) = \cos^2\theta \, g(X, Y) \tag{19}$$

$$g(FX, FY) = \sin^2\theta \, g(X, Y) \tag{20}$$

for any $X, Y \in \Gamma(TM)$.

Let M be a proper slant submanifold of an almost Hermitian manifold $\overline{M}$, then $FT_x M$ is subspace of $\mathrm{T}^{\perp}_X M$. Thus for any $x \in M$, we decopose the normal space as

$$T^{\perp}M = FTM \bigoplus \mu$$

where μ is an invariant subbundle under J orthogonal to FTM.

TOTALLY UMBILICAL PROPER SLANT SUBMANIFOLDS

Throughout the section, we assume $\overline{M}$ to be a nearly Keahler manifold and M be a submanifold of $\overline{M}$. Thus, on a submanifold $M$of a nearly Kaehler manifold $\overline{M}$, it follows from (2) that

$$(a)\ P_X Y + \ P_Y X = 0, \qquad (b)\ Q_X Y + \ Q_Y X = 0 \tag{21}$$

for any $X, Y \in \Gamma(TM)$. First, we assume that M is a totally umbilical submanifold of a nearly Kaehler manifold $\overline{M}$ in Theorem 3.1 and then we consider M as a totally umbilical proper slant submanifold of a nearly Kaehler manifold $\overline{M}$ to prove our main result.

Theorem 3.1. Let M be a totally umbilical submanifold of a nearly Kaehler manifold $\overline{\mathrm{M}}$. Then the following conditions are equivalent:

(i) The submanifold M has a nearly Kaehler structure (T, g)

(ii) $\mathrm{H} \in \Gamma(\mu)$

where H is the mean curvature vector on M.

Proof. As M is a totally umbilical submanifold then for any $X \in \Gamma(TM)$, we have

$$\mathrm{h(X, TX)} = \mathrm{g(X, TX)H} = 0.$$

Using (4) we obtain

$$\overline{\nabla}_{\mathrm{X}}\ \mathrm{TX} - \nabla_{\mathrm{X}}\mathrm{TX} = 0.$$

Then from (8), we get

$$\overline{\nabla}_X JX - \overline{\nabla}_X FX = \nabla_{\mathrm{X}}\mathrm{TX}.$$

Thus on using (3), we arrive at

$$\left(\overline{\nabla}_X J\right)X + \mathrm{J}\overline{\nabla}_X X - \overline{\nabla}_X FX = \nabla_{\mathrm{X}}\mathrm{TX}.$$

Using (2), (4) and (5), we get

$$J(\nabla_X X + h(X, X)) + A_{FX}\mathrm{X} - \nabla_X^{\perp}\mathrm{FX} = \nabla_{\mathrm{X}}\mathrm{TX}.$$

Then from (8) and (9), we obtain

$$F\nabla_X X + T\nabla_X X + th(X,X) + fh(X,X) + A_{FX}X - \nabla_X^{\perp}FX = \nabla_X TX. \tag{22}$$

Equating the tangential components, we get

$$T\nabla_X X + th(X,X) + fh(X,X) + A_{FX}X = \nabla_X TX. \tag{23}$$

As M is totally umbilical, the above equation takes the form

$$\left(\overline{\nabla}_X T\right)X = g(X,X)tH + g(H,FX)X. \tag{24}$$

The relation (24) has a solution that: if $H \in \Gamma(\mu)$, then $\left(\overline{\nabla}_X T\right)X = 0$ and vice-versa. This completes the proof of the theorem.

Theorem 3.2. Let M be a totally umbilical proper slant submanifold of a nearly Kaehler manifold $\overline{\mathrm{M}}$. Then M is totally geodesic with nearly Kaehler structure T.

Proof. For any $X, Y \in \Gamma(TM)$, we have

$$\left(\overline{\nabla}_X J\right)Y = \overline{\nabla}_X JY - J\overline{\nabla}_X Y.$$

Using (4) and (8), we get

$$P_X Y + Q_X Y = \overline{\nabla}_X TY + \overline{\nabla}_X FY - T\nabla_X Y - F\nabla_X Y - Jh(X,Y).$$

Again from (4) and (5), we obtain

$$P_X Y + Q_X Y = \nabla_X TY + h(X,TY) - A_{FY}X + \nabla_X^{\perp}FY - T\nabla_X Y - F\nabla_X Y - Jh(X,Y).$$

Then from (7), we derive

$$P_X Y + Q_X Y = \nabla_X TY + g(X,TY)H - A_{FY}X + \nabla_X^{\perp}FY - T\nabla_X Y - F\nabla_X Y - g(X,Y)JH. \tag{25}$$

Taking the product in (25) with JH and using the Theorem 3.1 (equivalent conditions), we deduce that

$$g(Q_X Y, JH) = g(X,TY)g(H,JH) + g\left(\nabla_X^{\perp}FY, JH\right) - g(F\nabla_X Y, JH) - g(X,Y)g(JH,JH).$$

Using (1) and again the Theorem 3.1 (equivalent conditions), we obtain

$$g(Q_X Y, JH) = g\left(\nabla_X^{\perp}FY, JH\right) - g(X,Y)\|H\|^2.$$

Thus by (3), we get

$$g(Q_X Y, JH) = \mathrm{g}(\overline{\nabla}_X \mathrm{FY}, \mathrm{JH}) - g(X, Y)\|H\|^2. \tag{26}$$

Similarly, we can obtain

$$g(Q_Y X, JH) = \mathrm{g}(\overline{\nabla}_Y \mathrm{FX}, \mathrm{JH}) - g(X, Y)\|H\|^2. \tag{27}$$

Adding (26) and (27) and on applying (21.b), we arrive at

$$g(\overline{\nabla}_X \mathrm{FY}, \mathrm{JH}) + \mathrm{g}(\overline{\nabla}_Y \mathrm{FX}, \mathrm{JH}) = 2g(X, Y)\|H\|^2. \tag{28}$$

Now for any $X \in \Gamma(TM)$, we have

$$\overline{\nabla}_X \mathrm{JH} = (\overline{\nabla}_X \mathrm{J})\mathrm{H} + \mathrm{J}\overline{\nabla}_X \mathrm{H}.$$

Using (5), (8) and (9) in this equation, we get

$$-A_{JH}X + \nabla^{\perp}_X \mathrm{JH} = \mathrm{P_X H} + \mathrm{Q_X H} - \mathrm{TA_H X} - \mathrm{FA_H X} + \mathrm{t}\nabla^{\perp}_X H + f\nabla^{\perp}_X H. \tag{29}$$

Taking the inner product in (29) with FY for any $Y \in \Gamma(TM)$, and using the fact that $f\nabla^{\perp}_X H \in \Gamma(\mu)$, the above equation gives

$$g(\nabla^{\perp}_X \mathrm{JH}, \mathrm{FY}) = -\mathrm{g}(\mathrm{FA_H X}, \mathrm{FY}) + \mathrm{g}(\mathrm{Q_X H}, \mathrm{FY}). \tag{30}$$

Then from (20), we get

$$\mathrm{g}(\nabla^{\perp}_X \mathrm{JH}, \mathrm{FY}) = -\sin^2\theta\ \mathrm{g}(\mathrm{A_H X}, \mathrm{FY}) + \mathrm{g}(\mathrm{Q_X H}, \mathrm{FY}).$$

Since $H \in \Gamma(\mu)$ (by Theorem 3.1), using (5) and (6), we obtain

$$\mathrm{g}(\overline{\nabla}_X \mathrm{FY}, \mathrm{JH}) = \sin^2\theta\ \mathrm{g}(\mathrm{h}(\mathrm{X}, \mathrm{Y}), \mathrm{H}) - \mathrm{g}(\mathrm{Q_X H}, \mathrm{FY}).$$

Thus from (7), we derive

$$g(\overline{\nabla}_X \mathrm{FY}, \mathrm{JH}) = \sin^2\theta\ \mathrm{g}(\mathrm{X}, \mathrm{Y})\|H\|^2 - \mathrm{g}(\mathrm{Q_X H}, \mathrm{FY}). \tag{31}$$

Similarly, we obtain

$$g(\overline{\nabla}_Y \mathrm{FX}, \mathrm{JH} = \sin^2\theta\ \mathrm{g}(\mathrm{X}, \mathrm{Y})\|H\|^2 - \mathrm{g}(\mathrm{Q_Y H}, \mathrm{FX}). \tag{32}$$

Adding (31) and (32), we get

$$g(\overline{\nabla}_X \mathrm{FY} + \overline{\nabla}_{\mathrm{Y}} \mathrm{FX}, \mathrm{JH}) = 2\sin^2\theta\ \mathrm{g}(\mathrm{X}, \mathrm{Y})\|H\|^2 - \mathrm{g}(\mathrm{Q_X H}, \mathrm{FY}) - \mathrm{g}(\mathrm{Q_Y H}, \mathrm{FX}). \tag{33}$$

Then from (28), we derive

$$2g(X,Y)\|H\|^2 = 2\sin^2\theta\ g(X,Y)\|H\|^2 - g(Q_X H, FY) - g(Q_Y H, FX).$$

That is

$$2\cos^2\theta\ g(X,Y)\|H\|^2 + g(Q_X H, FY) + g(Q_Y H, FX) = 0. \tag{34}$$

Now, from (17), we have

$$g(Q_X H, FY) = g\big((\overline{\nabla}_X f)H, FY\big) + g(h(tH, X)FY) + g(FA_H X, FY),$$

for any $X, Y \in \Gamma(TM)$. Using the equivalent conditions of Theorem 3.1 and then from (13) and (20), we obtain

$$g(Q_X H, FY) = \sin^2\theta\ g(A_H X, Y). \tag{35}$$

Then from (6) and (7), we get

$$g(Q_X H, FY) = \sin^2\theta\ g(X, Y)\|H\|^2 \tag{36}$$

Using this fact in (34), we obtain

$$g(X, Y)\|H\|^2 = 0. \tag{37}$$

It follows from (37) that $H = 0$, that is, M is totally geodesic. This proves the theorem completely.

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مذكرة حول منطوى جزئي فعلي مائل وكلي السُرِّية لمنطوى قريب - كايلر

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خلاصة

ندرس في هذا البحث منطوي جزئي فعلي مائل وكلي السُرِّية لمنطوي قريب - كايلر. ونثبت أن هذه المنطويات الجزئية يجب أن تكون كلية الجيوديزية.