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GENERALIZED ULAM–HYERS STABILITY FOR FRACTIONAL DIFFERENTIAL EQUATIONS

RABHA W. IBRAHIM

Institute of Mathematical Sciences, University Malaya
Kuala Lumpur 50603, Malaysia
rabhaibrahim@yahoo.com

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In the present paper, we consider the generalized Hyers–Ulam stability for fractional differential equations of the form:

$$D_z^\alpha f(z) = G(f(z), zf'(z), z^2 f''(z); z), \quad 2 < \alpha \leq 3$$

in a complex Banach space. Furthermore, applications are illustrated.

Keywords: Analytic function; unit disk; Hyers–Ulam stability; admissible functions; fractional calculus; fractional differential equation.

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1. Introduction

A classical problem in the theory of functional equations is that: If a function f approximately satisfies functional equation $\mathcal{E}$ when does there exists an exact solution of $\mathcal{E}$ which f approximates. In 1940, Ulam [28, 29] imposed the question of the stability of Cauchy equation and in 1941, Hyers solved it [5]. In 1978, Rassias [18] provided a generalization of Hyers, theorem by proving the existence of unique linear mappings near approximate additive mappings. The problem has been considered for many different types of spaces (see [6–8]). Recently, Li and Hua [14] discussed and proved the generalized Hyers–Ulam stability of spacial type of finite polynomial equation, and Bidkham, Mezerji and Gordji [1] introduced the Hyers–Ulam stability of generalized finite polynomial equation. Finally, Rassias [21] imposed a Cauchy type additive functional equation and investigated the generalized Hyers–Ulam product–sum stability of this equation. The stability phenomenon that was proved by Rassias is called the Ulam–Gavruta–Rassias stability [23]. Very recently Rassias [24] introduced a new concept on stability called Rassias Mixed type product–sum of powers of norms stability. Furthermore, Rassias [13, 19, 20, 22] introduced and investigated the stability problem of Ulam for the Euler–Lagrange quadratic mappings and relative Euler–Lagrange functional equation.

The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann–Liouville operators [3], Erdlyi–Kober operators [12], Weyl–Riesz operators [16], Caputo operators [2] and Grnwald–Letnikov operators [17], have appeared during the past three decades. The existence of positive solution and multi-positive solutions for nonlinear fractional differential equation are established and studied [30]. Moreover, by using the concepts of the subordination and superordination of analytic functions, the existence of analytic solutions for fractional differential equations in complex domain are suggested and posed in [9–11].

In [27], Srivastava and Owa, gave definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows:

Definition 1.1. The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta,$$

where the function $f(z)$ is analytic in simply connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.2. The fractional integral of order $\alpha > 0$ is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad \alpha > 0,$$

where the function $f(z)$ is analytic in simply connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark 1.1.

$$D_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \quad \mu > -1$$

and

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \quad \mu > -1.$$

In [11], it was shown the relation

$$I_z^\alpha D_z^\alpha f(z) = D_z^\alpha I_z^\alpha f(z) = f(z).$$

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}$ denote the space of all analytic functions on U. Here we suppose that $\mathcal{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of U. Also for $a \in \mathbb{C}$ and $m \in \mathbb{N}$, let $\mathcal{H}[a, m]$ be the subspace of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \cdots, \quad z \in U.$$

Definition 1.3. Let p be a real number. We say that

$$\sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z) \tag{1.1}$$

has the generalized Hyers–Ulam stability if there exists a constant $K > 0$ with the following property:

for every $\epsilon > 0, w \in \overline{U} = U \cup \partial U$, if

$$\left| \sum_{n=0}^{\infty} a_n w^{n+\alpha} \right| \leq \epsilon \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n} \right)$$

then there exists some $z \in \overline{U}$ that satisfies Eq. (1.1) such that

$$|z^i - w^i| \leq \epsilon K, \quad (z, w \in \overline{U}, i \in \mathbb{N}).$$

In the present paper, we study the generalized Hyers–Ulam stability for holomorphic solutions of the fractional differential equation in complex Banach spaces X and Y

$$D_z^\alpha f(z) = G(f(z), zf'(z), z^2 f''(z); z), \quad 2 < \alpha \leq 3, \tag{1.2}$$

where $G : X^3 \times U \to Y$ and $f : U \to X$ are holomorphic functions such that $f(0) = \Theta$ (Θ is the zero vector in X).

2. Generalized Hyers–Ulam Stability

In this section, we present extensions of the generalized Hyers–Ulam stability to holomorphic vector-valued functions. Let X, Y represent complex Banach space. The class of admissible functions $\mathcal{G}(X, Y)$, consists of those functions $g : X^3 \times U \to Y$ that satisfy the admissibility conditions.

$$\|g(r, ks, lt; z)\| \geq 1, \quad \text{when } \|r\| = 1, \ \|s\| = 1, \ \|t\| = 1, \quad (z \in U, k \geq 1, l \geq 1). \tag{2.1}$$

We need the following results.

Lemma 2.1 ([4]). *If $f : D \to X$ is holomorphic, then $\|f\|$ is a subharmonic of $z \in D \subset \mathbb{C}$. It follows that $\|f\|$ can have no maximum in D unless $\|f\|$ is of constant value throughout D.*

Lemma 2.2 ([25]). *Let $f : U \to X$ be holomorphic vector-valued function defined in the unit disk U with $f(0) = \Theta$ (the zero element of X). If there exists a $z_0 \in U$ such that*

$$\|f(z_0)\| = \max_{|z|=|z_0|} \|f\|,$$

then

$$\|z_0 f'(z_0)\| = k\|f(z_0)\|, \quad k \geq 1.$$

Theorem 2.1. *Let $g \in \mathcal{G}(X, Y)$. If $f : U \to X$ is the holomorphic vector-valued function defined in the unit disk U with $f(0) = \Theta$, then*

$$\left\|g(f(z), zf'(z), z^2 f''(z); z)\right\| < 1 \Longrightarrow \|f(z)\| < 1. \tag{2.2}$$

Proof. We assume $\|f(z)\| \not< 1$ for $z \in U$. Thus there exists a point $z_0 \in U$ with $\|f(z_0)\| = 1$. In view of Lemma 2.1,

$$\|f(z)\| < 1, \quad z \in U_r = \{z : |z| < |z_0| = r\}$$

and

$$\max_{|z|=|z_0|} \|f\| = \|f(z_0)\| = 1.$$

Now by Lemma 2.2, at the point z_0 the exists a constant $k \geq 1$ such that

$$\|z_0 f'(z_0)\| = k\|f(z_0)\| = k.$$

Consequently, we have

$$\left\|\frac{z_0 f'(z_0)}{k}\right\| = 1, \quad k \geq 1. \tag{2.3}$$

Furthermore, defining a function $\omega(z) := zf'(z)$ satisfies $\omega(0) = \Theta$ and

$$\|\omega(z_0)\| = \max_{|z|=|z_0|} \|\omega(z)\| = k.$$

Therefore, according to Lemma 2.2, there exists a constant $\kappa > 1$ such that

$$\|z_0 \omega'(z_0)\| = \kappa\|\omega(z_0)\|.$$

On the other hand, we observe that

$$\begin{aligned} \|z_0 \omega'(z_0)\| &= \|z_0^2 f''(z_0) + z_0 f'(z_0)\| \leq \|z_0^2 f''(z_0)\| + \|z_0 f'(z_0)\| \\ &= \|z_0^2 f''(z_0)\| + k\|f(z_0)\|, \end{aligned}$$

thus there exists a constant $m \geq 1$, such that

$$\|z_0^2 f''(z_0)\| = mk(\kappa - 1)\|f(z_0)\|,$$

consequently we pose that

$$\left\| \frac{z_0^2 f''(z_0)}{l} \right\| := \left\| \frac{z_0^2 f''(z_0)}{mk(\kappa - 1)} \right\| = 1, \quad k \geq 1, \ \kappa > 1, \ m \geq 1. \tag{2.4}$$

Combining Eqs. (2.3) and (2.4), we deduce

$$\|g(f(z_0), z_0 f'(z_0), z_0^2 f''(z_0); z_0)\| = \|g(f(z_0), k[z_0 f'(z_0)/k], l[z_0^2 f''(z_0)/l]; z_0)\| \geq 1,$$

this contradicts assumption (2.2), and we have $\|f(z)\| < 1$. This completes the proof of Theorem 2.1. □

Note that Theorem 2.1 is an extension of a result [15, Theorem 8.4c].

Corollary 2.1. *Assume the problem* (1.2). *Let* $G \in \mathcal{G}(X, Y)$ *be holomorphic vector-valued functions defined in the unit disk* U *then*

$$\|G(f(z), zf'(z), z^2 f''(z); z)\| < 1 \Longrightarrow \|I_z^{\alpha} G(f(z), zf'(z), z^2 f''(z); z)\| < 1. \tag{2.5}$$

Proof. By continuity of G, the fractional differential equation (1.2) has at least one holomorphic solution f. According to Remark 1.1, the solution $f(z)$ of the problem (1.2) takes the form

$$f(z) = I_z^{\alpha} G(f(z), zf'(z), z^2 f''(z); z).$$

Therefore, in virtue of Theorem 2.1, we obtain the assertion (2.5). □

Theorem 2.2. *Let* $G \in \mathcal{G}(X, Y)$ *be holomorphic vector-valued functions defined in the unit disk* U *then Eq.* (1.2) *has the generalized Hyers–Ulam stability for* $z \to \partial U$.

Proof. Assume that

$$G(z) := \sum_{n=0}^{\infty} \varphi_n z^n, \quad z \in U$$

therefore, by Remark 1.1, we have

$$I_z^{\alpha} G(z) = \sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z).$$

Also, $z \to \partial U$ and thus $|z| \to 1$. According to Corollary 2.1, we have

$$\|f(z)\| < 1 = |z|.$$

Let $\epsilon > 0$ and $w \in \overline{U}$ be such that

$$\left| \sum_{n=1}^{\infty} a_n w^{n+\alpha} \right| \leq \epsilon \left(\sum_{n=1}^{\infty} \frac{|a_n|^p}{2^n} \right).$$

We will show that there exists a constant K independent of ϵ such that

$$|w^i - u^i| \leq \epsilon K, \quad w \in \overline{U}, \ u \in U$$

and satisfies (1.1). We put the function

$$f(w) = \frac{-1}{\lambda a_i} \sum_{n=1, n\neq i}^{\infty} a_n w^{n+\alpha}, \quad a_i \neq 0, \quad 0 < \lambda < 1, \tag{2.6}$$

thus for $w \in \partial U$, we obtain

$$\begin{aligned}|w^i - u^i| &= |w^i - \lambda f(w) + \lambda f(w) - u^i| \\ &\leq |w^i - \lambda f(w)| + \lambda |f(w) - u^i| < |w^i - \lambda f(w)| + \lambda |w^i - u^i| \\ &= \left| w^i + \frac{1}{a_i} \sum_{n=1, n\neq i}^{\infty} a_n w^{n+\alpha} \right| + \lambda |w^i - u^i| = \frac{1}{|a_i|} \left| \sum_{n=1}^{\infty} a_n w^{n+\alpha} \right| + \lambda |w^i - u^i|.\end{aligned}$$

Without loss of the generality, we consider $|a_i| = \max_{n\geq 1}(|a_n|)$ yielding

$$\begin{aligned}|w^i - u^i| &\leq \frac{1}{|a_i|(1-\lambda)} \left| \sum_{n=1}^{\infty} a_n w^{n+\alpha} \right| \leq \frac{\epsilon}{|a_i|(1-\lambda)} \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n} \right) \\ &\leq \frac{\epsilon |a_i|^{p-1}}{(1-\lambda)} \left(\sum_{n=0}^{\infty} \frac{1}{2^n} \right) \leq \frac{2\epsilon |a_i|^{p-1}}{(1-\lambda)} := K\epsilon.\end{aligned}$$

This completes the proof. □

3. Applications

In this section, we introduce some applications of functions to achieve the generalized Hyers–Ulam stability.

Example 3.1. Consider the function $G : X^3 \times U \to \mathbb{R}$ by

$$G(r, s, t; z) = a(\|r\| + \|s\| + \|t\|) + b|z|,$$

with $a \geq 0.5$, $b \geq 0$ and $G(\Theta, \Theta, \Theta, 0) = 0$. Our aim is to apply Corollary 2.1. This follows since

$$\|G(r, ks, \kappa t; z)\| = a(\|r\| + k\|s\| + \kappa\|t\|) + b|z| = a(1 + k + \kappa) + b|z| \geq 1,$$

when $\|r\| = \|s\| = \|t\| = 1$, $z \in U$. Hence by Corollary 2.1, we have: If $a \geq 0.5$, $b \geq 0$ and $f : U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$a(\|f(z)\| + \|zf'(z)\| + \|z^2 f''(z)\|) + b|z| < 1 \Longrightarrow \|f(z)\| < 1.$$

Consequently, $\|I_z^\alpha G(f(z), zf'(z), z^2 f''(z); z)\| < 1$, thus in view of Theorem 2.2, f has the generalized Hyers–Ulam stability.

Example 3.2. Assume the function $G : X^3 \to X$ by

$$G(r, s, t; z) = G(r, s, t) = re^{\|s\|\|t\|-1},$$

with $G(\Theta, \Theta, \Theta) = \Theta$. By applying Corollary 2.1, we need to show that $G \in \mathcal{G}(X, X)$. Since

$$\|G(r, ks, \kappa t)\| = \|re^{\|ks\|\|\kappa t\|-1}\| = e^{k\kappa-1} \geq 1,$$

when $\|r\| = \|s\| = \|t\| = 1$, $k \geq 1$ and $\kappa > 1$. Hence by Corollary 2.1, we have: For $f: U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$\|f(z)e^{\|zf'(z)\|\|z^2f''(z)\|-1}\| < 1 \Longrightarrow \|f(z)\| < 1.$$

Consequently, $\|I_z^\alpha G(f(z), zf'(z), z^2f''(z); z)\| < 1$, thus in view of Theorem 2.2, f has the generalized Hyers–Ulam stability.

Example 3.3. Let $a, b, c: U \to \mathbb{C}$ satisfy

$$|a(z) + \mu b(z) + \nu c(z)| \geq 1,$$

for every $\mu \geq 1, \nu > 1$ and $z \in U$. Consider the function $G: X^3 \to Y$ by

$$G(r, s, t; z) = a(z)r + \mu b(z)s + \nu c(z)t,$$

with $G(\Theta, \Theta, \Theta) = \Theta$. Now for $\|r\| = \|s\| = \|t\| = 1$, we have

$$\|G(r, \mu s, \nu t; z)\| = |a(z) + \mu b(z) + \nu c(z)| \geq 1$$

and thus $G \in \mathcal{G}(X, Y)$. If $f: U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$\|a(z)f(z) + zb(z)f'(z) + z^2c(z)f''(z)\| < 1 \Longrightarrow \|f(z)\| < 1.$$

Hence according to Theorem 2.2, f has the generalized Hyers–Ulam stability.

4. Conclusion

From above, we conclude that the Hyers–Ulam stability can be generalized for fractional differential equations in a complex Banach space. Equation (1.1), referred to the general formula of approximate solutions for fractional differential equations in the unit disk, which can be extended to a complex Banach space. Definition 1.3, can be applied in the theory of analytic functions to find the characterization properties, coefficient estimates, distortion inequalities and convolution structures for various subclasses of analytic functions of fractional power [26].

Finally, we encourage (as open questions) the reader to investigate the Ulam–Gavruta–Rassias "product" stability and the Rassias "mixed product–sum" stability, as well as provide specific counter examples to the arising singular cases for fractional differential equation.

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