On cross-intersecting families of set partitions

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Abstract

Let $B(n)$ denote the collection of all set partitions of $[n]$. Suppose $A_1, A_2 \subseteq B(n)$ are cross-intersecting i.e. for all $A_1 \in A_1$ and $A_2 \in A_2$, we have $A_1 \cap A_2 \neq \emptyset$. It is proved that for sufficiently large $n$,

$$|A_1||A_2| \leq B^2_{n-1}$$

where $B_n$ is the $n$-th Bell number. Moreover, equality holds if and only if $A_1 = A_2$ and $A_1$ consists of all set partitions with a fixed singleton.

Keywords: cross-intersecting family, Erdős-Ko-Rado, set partitions

1 Introduction

1.1 Finite sets

Let $[n] = \{1, \ldots, n\}$ and $\binom{[n]}{k}$ denote the family of all $k$-subsets of $[n]$. A fundamental result in extremal combinatorial set theory is the Erdős-Ko-Rado theorem ([6], [7], [22]) which asserts that if a family $\mathcal{A} \subseteq \binom{[n]}{k}$ is $t$-intersecting (i.e. $|A \cap B| \geq t$ for any $A, B \in \mathcal{A}$), then $|\mathcal{A}| \leq \binom{n-t}{k-t}$ for $n \geq (k-t+1)(t+1)$. Recently, there are several Erdős-Ko-Rado type results (see [2, 4, 5, 9, 11, 13, 15, 17, 20, 21]), most notably is the result of Ellis, Friedgut and Pilpel [5], which states that for sufficiently large $n$ depending on $t$, a $t$-intersecting family $\mathcal{A}$ of permutations has size at most $(n-t)!$, with equality if and only if $\mathcal{A}$ is a coset of the stabilizer of $t$ points, thus settling an old conjecture of Deza and Frankl [3].