A Kruskal–Katona type theorem for integer partitions

Cheng Yeaw Ku\textsuperscript{a,}\textsuperscript{*}, Kok Bin Wong\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, National University of Singapore, Singapore 117543, Singapore
\textsuperscript{b} Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia

\begin{abstract}
Let \( \mathbb{N} \) be the set of positive integers, and let
\[
P(n) = \bigcup_{1 \leq x \leq n} \{(x_1, \ldots, x_t) \in \mathbb{N}^t : x_1 + \cdots + x_t = n\}
\]
be the set of (ordered) partitions of \( n \). We show that there exist a rank function and orderings \( \leq_c \) and \( < \) such that the ranked poset \( (P(n), \leq_c, <) \) is Macaulay.

© 2013 Elsevier B.V. All rights reserved.
\end{abstract}

1. Introduction

Let \((P, \leq_c)\) be a poset, and let \( x \) and \( y \) be elements in \( P \). We say that \( y \) covers \( x \) if there is no \( z \in P, z \not\in \{x, y\} \), such that \( x \leq z \leq y \). Let \( \mathbb{N} \) be the set of all positive integers. A poset \( P \) is called a ranked poset, if there is a rank function \( r : P \to \mathbb{N} \) such that \( r(y) = r(x) + 1 \) whenever \( y \) covers \( x \). We only consider ranked posets in this paper. The set of all elements of rank \( i \) is denoted by \( P_i \), i.e. \( P_i = \{x \in P : r(x) = i\} \). The shadow of an element \( x \in P \), denoted by \( \Delta(x) \), is defined by
\[
\Delta(x) = \{y \in P : y \leq_c x, r(y) = r(x) - 1\}.
\]
For a subset \( X \) of \( P \), its shadow \( \Delta(X) \) is defined by
\[
\Delta(X) = \bigcup_{x \in X} \Delta(x).
\]
We set \( \Delta(X) = \emptyset \) for any \( X \subseteq P_0 \). Given positive integers \( i \) and \( m \), where \( 1 \leq m \leq |P_i| \), the shadow minimization problem (SMP) consists of finding a subset \( X \) of \( P_i \) such that \( |X| = m \) and \( |
\Delta(X)| \leq |
\Delta(Y)\| \) for all \( Y \subseteq P_i \) with \( |Y| = m \). Solutions to the SMP are related to many combinatorial problems such as the isoperimetric problems on graphs and problems in polyhedral combinatorics. For details of such applications, the reader is referred to the survey [3] and references therein.

Let \( < \) be a linear ordering on the elements of \( P \). For any subset \( X \) of \( P_i \), let \( C(X) \) be the set of the first \( |X| \) elements of \( P_i \) with respect to \( (w.r.t.) < \). If \( C(X) = X \), then \( X \) is said to be compressed. The ranked poset \((P_i, \leq_c, <)\) is a Macaulay poset [8] if the following conditions are satisfied for all \( i \) and all subsets \( X \) of \( P_i \):

\begin{enumerate}
\item \( \Delta(C(X)) \) is compressed in \( P_{i-1} \),
\item \( |\Delta(C(X))| \leq |\Delta(X)| \).
\end{enumerate}

One of the most popular classes of Macaulay posets is that of Boolean lattices. Let \( 2^{[n]} \) denote the set of all subsets of \([n]\), where \([n] = \{1, \ldots, n\} \). The Boolean lattice \( B^n \) is the poset formed on the elements of \( 2^{[n]} \) partially ordered by inclusion, i.e. \( A \leq B \) if and only if \( A \subseteq B \). The rank function for this poset is given by \( r(A) = |A| \).

\* Corresponding author.
E-mail addresses: matky@nus.edu.sg (C.Y. Ku), kkwong@um.edu.my (K.B. Wong).

0012-365X/3 - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.disc.2013.06.001