Abstract—The Z-transform method for modeling the dispersive media is analyzed mathematically. The most complicated part of modeling of this medium is to find a relationship between the flux densities and the field intensities. Because of the simplicity of the finite difference in time-domain, this relationship can be looked upon as a digital filtering issue. In this paper, the Z-transform is used to incorporate the dispersive medium constitutive relations into the frequency algorithms. An IIR filter structure is designed to show the non-linear behavior of the frequency dependent permeability of materials using properties of the Z-transform.

Index Terms—Dispersive media; finite-difference time-domain, Z-transform, IIR filter

I. INTRODUCTION

The finite-difference time-domain (FDTD) method has played a very important role in the analysis of electromagnetic radiation for a broad range of applications [1]–[3]. Many approaches are introduced for modeling dispersive media using FDTD method. Several summaries of various methods are given in [4] and [5]. This FDTD formulation is a direct implementation of the coupled constitutive relations incorporated into Maxwell’s equations, which use the decoupled equations. Dispersive materials have been realized by using different models such as the cold plasma, Debye, Drude and Lorentz models. Material dispersion is modeled by the Z-transform method [6]. However, the Z-transform approach has the best accuracy near resonant frequencies and is very easy to implement.

In this transform, backward time differences are used for time derivatives on order to decouple updating equations for electric and magnetic field-inked components. Pereda et al. [7] expressed the Maxwell’s curl equations in Laplace domain and then directly transformed them to the Z-domain. In the present paper, the FDTD formulation for dispersive materials is extended based on Debye, Drude and Lorentz approximations using the Z-transform method. Furthermore, a digital filter structure based on infinite impulse response (IIR) is proposed to calculate the field densities in terms of field intensities quickly. In Section II, the theoretical framework of the dispersive media formulation using Z-transform is presented. A convolution based solution of electromagnetic fields is developed. Section III gives a discussion about the derived results and validates our mathematical algorithms. Finally, conclusions are included in Section IV.

II. METHODOLOGY

In the FDTD method, a description of how this method is applied to the Maxwell’s equations via Z-transform is considered. The frequency-domain Maxwell’s equation can be written as

$$\nabla \times E(\omega) = -j\omega\mu(\omega)H(\omega)$$  \hspace{1cm} (1a)
$$\nabla \times H(\omega) = j\omega\varepsilon(\omega)E(\omega)$$  \hspace{1cm} (1b)
$$\nabla \cdot D(\omega) = \rho_e$$  \hspace{1cm} (1c)
$$\nabla \cdot B(\omega) = \rho_m$$  \hspace{1cm} (1d)

where $\rho_e$ and $\rho_m$ are the electric and magnetic charge densities, respectively. For the electric and magnetic flux densities of dispersive material, the frequency domain constitutive equations can be described as

$$D(\omega) = \varepsilon(\omega)E(\omega)$$  \hspace{1cm} (1a)
$$B(\omega) = \mu(\omega)H(\omega)$$  \hspace{1cm} (1b)

A. Dispersive models in frequency domain

1) Debye model: For a first-order Debye medium the frequency dependent permeability is given by

$$\mu(\omega) = \mu_0 \frac{1}{1 - j\omega\tau} = \frac{B_z(\omega)}{H_z(\omega)}$$  \hspace{1cm} (2)

where $\mu_0$ is the free space permeability and $\tau$ is the Debye relaxation time constant. $\alpha$ is a positive factor. The dissipativity of the material is seen from the fact that the imaginary part of $\mu(\omega)$ is negative for all frequencies (the same situation is happened for $\varepsilon(\omega)$). The imaginary part of $\mu(\omega)$ has its minimum at the relaxation frequency $f_r = \frac{1}{2\pi\tau}$ and at this frequency, we get:

$$\text{Im}\{\mu(\omega_r)\} = \frac{\alpha\mu_0}{2} ; \text{Re}\{\mu(\omega_r)\} = \mu_0(1 + \frac{\alpha}{2})$$  \hspace{1cm} (3)

Taking the inverse Fourier transform of (2) gives us the time-dependant permeability as

$$\mu(t) = \mu_0\delta(t) - \frac{\alpha\mu_0}{\tau} e^{\tau t} u(t)$$  \hspace{1cm} (4)

A partial differential equation based on (2), can be related $B_z$ and $H_z$ as,

$$\frac{\partial}{\partial t}(\mu_0H_z - \tau B_z) = (1 + \alpha)\mu_0H_z - B_z$$  \hspace{1cm} (5)
Equation (5) can be solved as the following convolution by using (4):

\[ B_z(t) = \mu_0 H_z(t) - \frac{\alpha \mu_0}{\tau} \int_0^\infty e^{\lambda/\tau} H_z(t-\lambda) d\lambda \]  

(6)

2) **Drude model:** For a lossy Drude polarization and magnetization models of a dispersive medium; specifically the permittivity and permeability are described in the frequency domain as

\[ \epsilon(\omega) = \epsilon_0 \left( 1 - \frac{\omega_{pe}^2}{\omega(\omega-j\Gamma_e)} \right) \]  

(6a)

\[ \mu(\omega) = \mu_0 \left( 1 - \frac{\omega_{pm}^2}{\omega(\omega-j\Gamma_m)} \right) \]  

(6b)

where \( \omega_{pe}, \omega_{pm} \) and \( \Gamma_e, \Gamma_m \) denote the corresponding plasma and damping frequencies, respectively, \( \epsilon_0 \) and \( \mu_0 \) are the free-space parameters. The typical conductivity behavior for low frequencies \( \epsilon(\omega) \approx \frac{\mu_0 \omega_{pe}}{\omega} \) is apparent in this model, where the conductivity is \( \alpha = \frac{\mu_0 \omega_{pe}}{\omega} \). The frequency domain equations described in (6a) must be expressed in the time domain to be compatible with the FDTD paradigm. Based on the convolution theory, we get

\[ D(t) = \epsilon_0 E(t) + \int_0^\infty (1-e^{-T/\tau}) E(t-\lambda) d\lambda \]  

(6a)

\[ B(t) = \mu_0 H(t) + \int_0^\infty (1-e^{-T/\tau}) H(t-\lambda) d\lambda \]  

(6b)

3) **Lorentz model:** The Lorentz model is a widely used second-order model in solid-state physics, and it predicts the frequency dependence of the permeability function as

\[ \mu(\omega) = \mu_0 + \frac{\omega_{pm}^2}{\omega^2 + j\omega\varsigma - \omega^2} \]  

(7)

where \( \omega_0 \) and \( \varsigma \) are the resonance and damping frequencies, respectively. By taking the inverse Fourier transform of (7), we can get the time-dependent permeability as

\[ \mu(t) = \mu_0 \delta(t) + \frac{\omega_{pm}^2}{\omega_0^2} \sin(\varsigma_0 t) e^{-\varsigma t/2} \]  

(8)

where \( \varsigma_0^2 = \omega_0^2 - (\varsigma/2)^2 \).

B. **Dispersive models in Z-domain**

In the previous Subsection, we reviewed three frequency models of the basic materials and also we obtained the relationships of the electric and magnetic fields intensities with their corresponding field densities using convolution formula. Finding the proper solutions to these convolutions have been the key to implement the in Z-domain. For example, the Z-transform of (2) in the Debye model is

\[ \mu(z) = \frac{\mu_0}{T} - \frac{\alpha \mu_0}{\tau} \frac{1}{1-z^{-1} e^{-\tau/\tau}} \]  

(9)

where \( T \) is the sampling period of time discretization in Z-transform. Fig. 1 shows the first-order IIR filter structure to implement Debye model in the Z-domain. Block D indicates the unit delay. The same implementation of the Drude dispersive medium can be represented by the following transform:

\[ \mu(z) = \frac{\mu_0}{T} + \frac{\mu_0 \omega_{pm}^2}{\Gamma_m} \left( \frac{1}{1-z^{-1}} - \frac{1}{1-z^{-1} e^{-\tau/\tau}} \right) \]  

(10)

Fig. 1: First-order IIR filter implementation of Debye model

By using (10) to describe the discrete behavior of (6b), we can design a second-order IIR filter to implement the Drude model of permeability function which is shown in Fig. 2. Finally, for the Lorentz model which is included two conjugate poles, the Z-transform version of (7) can be expressed as

\[ \mu(z) = \frac{\mu_0}{T} + \left[ \frac{\omega_{pm}^2}{\omega_0} \right] K_1 z^{-1} - \frac{1}{1-2K_2 z^{-1} + K_3 z^{-2}} \]  

(11)

where \( K_1 = e^{-\varsigma T/2} \sin(\varsigma_0 T) \), \( K_2 = e^{-\varsigma T/2} \cos(\varsigma_0 T) \) and \( K_3 = e^{-\varsigma T} \). Fig. 3 illustrates the implementation of the Lorentz model via a second-order IIR filter structure.

Fig. 2: Second-order IIR filter implementation of Drude model

III. **DISCUSSION**

To show the validity of the proposed formulation and IIR filter structures to calculate the dispersive media parameters, a comparison of the Z-domain method with commonly used frequency domain is performed. For a first-order Debye model
Fig. 3: Second-order IIR filter implementation of Lorentz model

with the parameters of $\tau = 0.1\,\text{s}$ and $\alpha = 7.5 \times 10^6$, the real and imaginary parts of the permeability function are plotted in Fig. 4 using both frequency and Z-domains approaches. From this figure, it is clear that the IIR filter implementation of permeability function is a good approximation in discrete-time space. Fig. 5 shows the permeability calculations of a second-order Drude model including two poles (one at origin) with the parameters of $\omega_{pm} = 1.7889 \times 10^5\,(\text{rad/s})$ and $\Gamma_m = 3.5\pi \times 10^3$. The permeability function of a Lorentz model with a pair of complex conjugate poles is shown in Fig. 6. The parameters of this model are: $\omega_{pm} = 5.0133 \times 10^{13}\,(\text{rad/s})$, $\omega_0 = 4 \times 10^{16}\,(\text{rad/s})$ and $\varsigma = 0.56 \times 10^{16}$.

IV. CONCLUSION

In this paper, formulations of the dispersive media models are presented. The main motivation is that the complicated convolution integrals in time-domain can be reduced to an algebraic equation in Z-domain. Based in the three famous model of dispersion theory, which are Debye, Drude and Lorentz, three IIR filter structures are designed to calculate the dispersion parameters.

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