A class of bivariate negative binomial distributions with different index parameters in the marginals

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Abstract

In this paper, we consider a new class of bivariate negative binomial distributions having marginal distributions with different index parameters. This feature is useful in statistical modelling and simulation studies, where different marginal distributions and a specified correlation are required. This feature also makes it more flexible than the existing bivariate generalizations of the negative binomial distribution, which have a common index parameter in the marginal distributions. Various interesting properties, such as canonical expansions and quadrant dependence, are obtained. Potential application of the proposed class of bivariate negative binomial distributions, as a bivariate mixed Poisson distribution, and computer generation of samples are examined. Numerical examples as well as goodness-of-fit to simulated and real data are also given here in order to illustrate the application of this family of bivariate negative binomial distributions.

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1. Introduction

The univariate negative binomial distribution has the following probability mass function:

$$r(x) = \frac{(\nu)^x}{x!} p^x (1 - p)^\nu \quad (0 < p < 1; \nu > 0; \quad x = 0, 1, 2, \ldots).$$

where $p$ is the probability parameter and $\nu$ is the index (dispersion) parameter. The negative binomial distribution has important applications in various disciplines [17]. Thus, clearly, there are a number of univariate generalizations (see, for example, Gupta and Ong [14]) and bivariate extensions of the negative binomial distribution [9,48]. Kocherlakota and Kocherlakota [22] and Mardia [31] have reviewed these bivariate negative binomial distribution as well as other bivariate generalizations of important univariate discrete distributions. Bivariate generalizations of important univariate distributions are of continuing interest (see, for example, Kundu and Gupta [24]). These bivariate generalizations can be constructed in a wide variety of ways like mixing or compounding, sampling and trivariate reduction. The method of trivariate reduction or random element in common is a popular method of construction due to its simplicity and ease of generating samples on a computer given a univariate generator. This method is described as detailed below.

Given independent random variables $Y_1$, $Y_2$ and $W$ from the same family, a bivariate generalization $(X_1, X_2)$ is given by

$$X_1 = Y_1 + W \quad \text{and} \quad X_2 = Y_2 + W.$$  (1)

However, this method of genesis imposes a restrictive range on the correlation coefficient $\rho$.  

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Apart from having a restricted range of the correlation coefficient, bivariate extensions of the negative binomial distribution, which are found in the existing literature, also suffer from a lack of flexibility due to the marginal distributions having the same index parameter $\nu$ [9,32,48]. In many Monte Carlo simulation experiments, there is a need to vary the degree of dependence [7, p. 573] or specify different marginal distributions especially when the bivariate structure of the distribution is not well understood. Clearly, the possession of different marginal distributions makes the bivariate distribution more flexible for empirical modelling.

In this paper, we introduce a class of bivariate negative binomial distributions based upon an extension of trivariate reduction, which have different marginal distributions. The proposed bivariate negative binomial distribution may be formulated as a mixed Poisson model. Mixed Poisson models form a useful class of distributions in practical applications (see, for example, Johnson et al. [17]). Applications of bivariate mixed Poisson distributions have been examined by (among others) Stein and Juritz [47], Aitchison and Ho [1] and Chib and Winkelmann [5]. Edwards and Gurland [9] and Subrahmaniam [48] have considered their bivariate negative binomial distributions in modelling accident proneness in the context of the bivariate Poisson model. We examine the potential application of the proposed extended bivariate negative binomial distribution, as a bivariate mixed Poisson distribution and computer generation of samples, as well as its application to data fitting. Various interesting properties, such as canonical expansion and quadrant dependence, will also be considered.

Quadrant dependence [29] is a useful concept of bivariate dependence which is easier to verify than the usual linear dependence. As special cases, the bivariate negative binomial distribution of Edwards and Gurland [9] and Subrahmaniam [48] are shown to be quadrant dependent.

A canonical expansion of a bivariate distribution is a single series expansion in terms of its marginal distributions and the corresponding orthogonal polynomials. The class of bivariate distributions formulated from the extension of trivariate reduction is shown to have canonical expansions. This extends the result of Eagleson [8]. For a particular case of the proposed extended bivariate negative binomial distribution, the explicit canonical expansion of the joint probability distribution is derived. The canonical expansion of a bivariate probability density function is a useful tool in the study of the structure of bivariate distributions [23]. Recently, Cuadras [6] derived canonical expansion in terms of distribution functions. Pertinent theory of canonical expansion of bivariate probability density function is outlined in the Appendix. As an example, the bivariate negative binomial distribution of Edwards and Gurland [9] and Subrahmaniam [48], with the joint probability generating function given by

$$G(z_1, z_2) = \sum_{x,y} z_1^x z_2^y h(x,y) = \left( \frac{\Theta}{1 - \theta_1 z_1 - \theta_2 z_2 - \theta_3 z_1 z_2} \right)^v,$$

$$(\Theta = 1 - \theta_1 - \theta_2 - \theta_3; \ \theta_i \geq 0 \ (i = 1, 2, 3); \ \Theta \leq 1)$$

and the joint bivariate probability mass function given by

$$h(x,y) = \Theta^\nu \sum_{i=0}^{\min(x,y)} \frac{\Gamma(v+x+y-i)\Gamma(v)\Gamma(y)\Gamma(x)\Gamma(\theta_1 \nu + i\theta_1^2)\Gamma(\theta_2 \nu + i\theta_2^2)\Gamma(\theta_3 \nu + i\theta_3^2)}{i!^2 i! i!}$$

has canonical (diagonal) expansion of the bivariate probability mass function [37] given by

$$h(x,y) = f(x)g(y) \sum_{i=0}^{\infty} \rho^i \mu^i \left( x; \nu, \frac{\theta_1 + \theta_2}{1 - \theta_2} \right) \mu^i \left( y; \nu, \frac{\theta_2 + \theta_3}{1 - \theta_1} \right),$$

where

$$f(x) = \frac{(v)_x}{x!} \left( \frac{\Theta}{1 - \theta_1} \right)^{\nu \left( \frac{\theta_1 + \theta_2}{1 - \theta_2} \right)} x^\nu,$$

$$g(y) = \frac{(v)_y}{y!} \left( \frac{\Theta}{1 - \theta_2} \right)^{\nu \left( \frac{\theta_2 + \theta_3}{1 - \theta_1} \right)} y^\nu,$$

and

$$\mu^i(x; \nu, \rho) = \frac{m_i(x; \nu, \rho)}{(v)_i i! \rho^i},$$

with the ith orthonormal Meixner polynomial defined by

$$m_i(x; \nu, \rho) = (v)_{i+1} F_1 \left( -i, -x; \nu; 1 - \frac{1}{\rho} \right),$$

in terms of the Gauss hypergeometric function $2F_1(a,b;c;z)$ defined by

$$2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} \quad (|z| < 1).$$
Here, and in what follows, $(\lambda)_n$ denotes the Pochhammer symbol defined, in terms of the familiar Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (v = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (v = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

$\mathbb{N}$ and $\mathbb{C}$ being, as usual, the sets of positive integers and complex numbers, respectively, it being understood conventionally that $(0)_0 := 1$.

This paper is organized as follows. In Section 2, an extension of the bivariate negative binomial distribution is proposed and formally defined, which is based upon an extension of the method of trivariate reduction and as a bivariate mixed Poisson distribution. The multivariate extension is also given. Distributional properties are considered in Section 3. A result extends the result of Eagleson [8] for the Meixner class of distributions. The canonical expansion of the extended bivariate negative binomial distribution is then derived. In Section 5, quadrant dependence of the extended bivariate negative binomial distribution is considered. Section 6 discusses the computer generation of bivariate samples with varying dependence. Applications and numerical examples as well as goodness-of-fit are considered in Section 7. Several concluding remarks and observations are presented in Section 8.

2. A new bivariate negative binomial distribution

2.1. Extended trivariate reduction

The method of trivariate reduction given by (1) may be extended by considering

$$X_1 = Y_1 + W_1 \quad \text{and} \quad X_2 = Y_2 + W_2,$$

(2)

where $(W_1,W_2)$ is a pair of randomly correlated elements independent of $Y_1$ and $Y_2$. Lai [25] examined the situation where

$$W_1 = I_1 W_1 \quad \text{and} \quad W_2 = I_2 W_2,$$

$I_1$ and $I_2$ being the indicator random variables such that $(I_1,I_2)$ has a joint probability distribution $(p_{00},p_{01},p_{10},p_{11})$.

For the genesis by (2), the general form of the bivariate probability generating function is given by

$$G_{\{X_1,X_2\}}(z_1,z_2) = G_{Y_1}(z_1; x_1, p_1)G_{Y_2}(z_2; x_2, p_2) \cdot G_{\{W_1,W_2\}}(z_1,z_2; v, \theta_1, \theta_2, \theta_3),$$

(3)

where $G_{Y_1}, G_{Y_2}$ and $G_{\{W_1,W_2\}}$ are the corresponding probability generating functions of $Y_1, Y_2$ and $(W_1, W_2)$. Furthermore, $Y_1, Y_2, W_1$ and $W_2$ are from the same family of univariate distributions with $(W_1, W_2)$ being jointly distributed. We note that, if $(W_1, W_2)$ has a distribution formulated by trivariate reduction, then $(X_1, X_2)$ again has a distribution formed by the usual trivariate reduction.

We now consider the extension of the bivariate negative binomial distribution when $(W_1, W_2)$ is distributed as the bivariate negative binomial distribution of Edwards and Gurland [9].

2.1.1. Extension of bivariate negative binomial distribution

Let

$$Y_1 \sim \text{NB}(x_1, p_1) \quad \text{and} \quad Y_2 \sim \text{NB}(x_2, p_2)$$

denote negative binomial random variables and let

$$(W_1, W_2) \sim \text{BNB}(v, \theta_1, \theta_2, \theta_3)$$

be the Edwards–Gurland bivariate negative binomial or compound correlated bivariate Poisson random variable with the joint probability generating function given by

$$G_{\{W_1,W_2\}}(z_1,z_2) = \left( \frac{\Theta}{1 - \theta_1 z_1 - \theta_2 z_2 - \theta_3 z_1 z_2} \right)^v (\Theta = 1 - \theta_1 - \theta_2 - \theta_3).$$

We note that this bivariate negative binomial distribution has a correlation coefficient $\rho$ in the range $0 \leq \rho \leq 1$. When $\theta_3 = 0$, this reduces to the bivariate negative binomial distribution which may be formulated through inverse sampling (see Kocherlakota and Kocherlakota [22, p. 122]).

By (3), this extended bivariate negative binomial distribution of $(X_1, X_2)$ has joint probability generating function

$$G_{\{X_1,X_2\}}(z_1,z_2) = \left( \frac{q_1}{1 - p_1 z_1} \right)^{x_1} \left( \frac{q_2}{1 - p_2 z_2} \right)^{x_2} \cdot \left( \frac{\Theta}{1 - \theta_1 z_1 - \theta_2 z_2 - \theta_3 z_1 z_2} \right)^v,$$

(4)

where

$$p_i = \frac{\theta_1 + \theta_3}{1 - \theta_2}, \quad p_i = \frac{\theta_2 + \theta_3}{1 - \theta_1}, \quad q_i = 1 - p_i \quad (i = 1, 2)$$
and
\[ \Theta = 1 - \theta_1 - \theta_2 - \theta_3, \]

with marginal probability generating functions of \( X_1 \) and \( X_2 \) given by
\[ G_{X_i}(z) = \left( \frac{q_i}{1 - p_i z} \right)^{z_i + \psi} \quad (i = 1, 2), \]

that is,
\[ X_1 \sim NB(z_1 + v, p_1) \quad \text{and} \quad X_2 \sim NB(z_2 + v, p_2). \]

This extended bivariate negative binomial has a correlation in the range 0 \( \leq \rho \leq 1 \) as seen from the Eq. (12), because the term \((\theta_3 + \theta_1 \theta_2)\) is positive.

### 2.1.2. Multivariate extensions

The following formulation:
\[ X_1 = Y_1 + W_1 \quad \text{and} \quad X_2 = Y_2 + W_2 \]

can be extended to develop multivariate distributions. By taking
\[ X_1 = Y_1 + W_1, \quad X_2 = Y_2 + W_2, \ldots, \quad X_k = Y_k + W_k, \]

a multivariate negative binomial distribution (MNB) with different parameters in the marginal is obtained as follows.

Let
\[ X_1 = Y_1 + W_1, \quad X_2 = Y_2 + W_2, \ldots, \quad X_k = Y_k + W_k, \]

where \( Y_1, Y_2, \ldots, Y_k \) are independent negative binomial random variables and
\[ W = (W_1, W_2, \ldots, W_k) \sim MNB, \]

with the probability generating function given by
\[ G_W(z_1, z_2, \ldots, z_k) = \left( \frac{\Theta}{1 - \sum_{i=1}^k \theta_i z_i - \sum_{1 \leq i < j \leq k} \theta_i z_i z_j - \cdots - \theta_{12} z_1 z_2 \cdots z_k} \right)^v, \]

where
\[ \Theta = 1 - \sum_{i=1}^k \theta_i - \sum_{1 \leq i < j \leq k} \theta_i z_i z_j - \cdots - \theta_{12} \cdot k. \]

We thus find that
\[ X = (X_1, X_2, \ldots, X_k) \sim MNB \]

has its probability generating function given by
\[ G_X(z_1, z_2, \ldots, z_k) = \prod_{i=1}^k \left( \frac{q_i}{1 - p_i z_i} \right)^{z_i + \psi} \left( \frac{\Theta}{1 - \sum_{i=1}^k \theta_i z_i - \sum_{1 \leq i < j \leq k} \theta_i z_i z_j - \cdots - \theta_{12} \cdot k z_1 z_2 \cdots z_k} \right)^v, \]

with the marginal probability generating functions given by
\[ G_{X_i}(z) = \left( \frac{q_i}{1 - p_i z} \right)^{z_i + \psi}, \]

where
\[ \phi_i = \theta_i + \sum_{j \neq i}^k \theta_j + \sum_{1 \leq i < j \leq k} \theta_{ij} + \cdots + \theta_{12} \cdot k, \quad p_i = \frac{\phi_i}{\Theta + \phi_i} \quad \text{and} \quad q_i = 1 - p_i \quad (i = 1, 2, \ldots, k). \]

Therefore, each of the marginals
\[ X_i \sim NB(z_i + v, p_i) \quad (i = 1, 2, \ldots, k) \]

will have different parameters.
2.2. Formulation as a mixed Poisson distribution

\[(X_1, X_2)\] is said to have a mixed Poisson distribution if its joint probability mass function is given by

\[
h(x_1, x_2; \xi) = \int f_x(x_1|\theta)f_x(x_2|\theta)g(\theta; \xi)d\theta,
\]

where \(f_x(x|\theta)\) is the Poisson probability mass function with parameter \(\theta\), and where \(\theta\) is a random variable \(\Theta\) with mixing distribution having probability density function \(g(\theta; \xi)\) with a vector of parameters \(\xi\). An extension of this is provided by

\[
h(x_1, x_2; \xi) = \int \int f_x(x_1|\theta_1)f_x(x_2|\theta_2)g(\theta_1, \theta_2; \xi)d\theta_1 d\theta_2,
\]

where \(\theta_1\) and \(\theta_2\) are jointly distributed with probability density function \(g(\theta_1, \theta_2; \xi)\) (see, for example, Ong [36]). We show that the extended bivariate negative binomial distribution can be formulated as a mixed Poisson distribution given by (5).

The bivariate gamma distribution considered by Gupta [11] has its moment generating function given by

\[
\text{Suppose that } X_1 \sim \text{Gamma}(\alpha_1, \beta_1) \quad \text{and} \quad Y_2 \sim \text{Gamma}(\alpha_2, \beta_2)\]

are independent gamma random variables with

\[\alpha_i - \beta > 0 \quad (i = 1, 2)\]

and

\[(W_1, W_2) \sim \text{Beta}(\gamma, \beta_1, \beta_2)\]

has the Wicksell–Kibble bivariate gamma distribution [21].

Let \(X \sim \text{Poi}(\lambda)\) denote the Poisson random variable with parameter \(\lambda\). The mixed Poisson formulation is given as follows. Suppose that

\[X_1|U \sim \text{Poi}(U) \quad \text{and} \quad X_2|V \sim \text{Poi}(V),\]

where \(U\) and \(V\) have a joint bivariate gamma distribution given by (6). Then, clearly, the unconditional \((X_1, X_2)\) has the extended bivariate negative binomial distribution.

The formulation is easily proved by using the following relation between the moment generating function of the mixing distribution and the probability generating function of the mixed distribution (see Ong [34]):

\[
G_{(X_1, X_2)}(z_1, z_2) = M_{(U,V)}(z_1 - 1, z_2 - 1).
\]

This leads to the extended bivariate negative binomial probability generating function given by

\[
G_{(X_1, X_2)}(z_1, z_2) = \left( \frac{q_1}{1 - p_1 z_1} \right)^{z_1 - \gamma} \left( \frac{q_2}{1 - p_2 z_2} \right)^{z_2 - \gamma} \left( \frac{\Theta}{1 - \theta_1 z_1 - \theta_2 z_2 - \theta_3 z_1 z_2} \right)^{\gamma},
\]

where

\[
p_i = \frac{1}{1 + \beta_i} \quad \text{and} \quad q_i = 1 - p_i \quad (i = 1, 2),
\]

\[
\theta_1 = p_1(1 - \rho^2 p_2), \quad \theta_2 = p_2(1 - \rho^2 p_1), \quad \theta_3 = - \frac{p_1 p_2 (1 - \rho^2)}{1 - \rho^2 p_1 p_2}
\]

and

\[
\Theta = 1 - \theta_1 - \theta_2 - \theta_3 = \frac{q_1 q_2}{1 - \rho^2 p_1 p_2}.
\]

We note that

\[-1 < \theta_3 < 0 \quad \text{with} \quad \theta_3 + \theta_1 \theta_2 > 0.\]

Rewriting in terms of \(\theta_1, \theta_2\) and \(\theta_3\), we have

\[
p_1 = \frac{\theta_1 + \theta_3}{1 - \theta_2} \quad \text{and} \quad p_2 = \frac{\theta_2 + \theta_3}{1 - \theta_1}.
\]
The marginal probability generating functions of \( X_1 \) and \( X_2 \) are given by
\[
G_{X_1}(z_1) = \left( \frac{q_1}{1 - p_1 z_1} \right)^{x_1} \quad \text{and} \quad G_{X_2}(z_2) = \left( \frac{q_2}{1 - p_2 z_2} \right)^{x_2},
\]
which yields
\[
X_1 \sim NB(x_1, p_1) \quad \text{and} \quad X_2 \sim NB(x_2, p_2).
\]
As a special case of this extended bivariate negative binomial distribution, the bivariate negative binomial distribution is included with \((U,V)\) having the Wicksell–Kibble bivariate gamma distribution (see Ong [34]), that is, when
\[
x_1 = x_2 = v
\]
in (6) and (7).

2.3. A general form of extended bivariate negative binomial

In the mixed Poisson formulation, it is found that \(-1 < \theta_3 < 0\) whereas by formulation (2), \(0 \leq \theta_3 < 1\). Hence, an extended bivariate negative binomial distribution with \(-1 < \theta_3 < 1\) can be now defined.

**Definition.** The joint probability generating function of the extended bivariate negative binomial distribution is given by
\[
G_{(X_1, X_2)}(z_1, z_2) = \left( \frac{q_1}{1 - p_1 z_1} \right)^{x_1} \left( \frac{q_2}{1 - p_2 z_2} \right)^{x_2} \left( \frac{\Theta}{1 - \theta_1 z_1 - \theta_2 z_2 - \theta_3 z_1 z_2} \right)^{v},
\]
with the following parameters:
\[
p_1 = \frac{\theta_1 + \theta_3}{1 - \theta_2}, \quad p_2 = \frac{\theta_2 + \theta_3}{1 - \theta_1}, \quad q_i = 1 - p_i \quad (i = 1, 2)
\]
and
\[
\Theta = 1 - \theta_1 - \theta_2 - \theta_3
\]
and the following restrictions:
\[
0 < p_1, p_2, \theta_1, \theta_2, \Theta < 1,
\]
\[-1 < \theta_3 < 1, \quad \theta_3 + \theta_1 > 0, \quad \theta_3 + \theta_2 > 0,
\]
\[-\theta_3 + \theta_1 \theta_2 > 0 \quad \text{and} \quad x_1, x_2, v > 0.
\]

This is obtained by combining (4) and (7) (by substituting \( x_1 - v \) for \( x_1 \) and \( x_2 - v \) for \( x_2 \)). We note that the marginal distributions
\[
X_1 \sim NB(x_1 + v, p_1) \quad \text{and} \quad X_2 \sim NB(x_2 + v, p_2)
\]
have different parameters. The correlation is in the range \((0,1)\) as indicated by Eq. (12).

3. Distributional properties

This section discusses the distributional properties for the extended bivariate negative binomial distribution obtained through the extended trivariate reduction and mixed Poisson formulations.

3.1. Joint probability mass function

In order to obtain the probability mass function of the extended bivariate negative binomial distribution from the corresponding probability generating function, the second member of Eq. (4) is expanded in powers of \( z_1 \) and \( z_2 \). The coefficient for the term \( z_1^{x_1} z_2^{x_2} \) will give the probability mass function denoted by
\[
Pr_X(x_1, x_2) = f(x_1, x_2)
\]
as follows:
\[
f(x_1, x_2) = f_{x_1}(x_1) f_{x_2}(x_2) \left( \frac{\Theta}{q_1 q_2} \right)^v \sum_{r=0}^{x_1} \sum_{s=0}^{x_2} \binom{x_1}{r} \binom{x_2}{s} \binom{z_1}{r} \binom{z_2}{s} \prod_{i=0}^{\min(r,s)} \binom{r}{i} \binom{s}{i} \left( \frac{\theta_3}{\theta_1 \theta_2} \right)^i,
\]
\((x_1, x_2 = 0, 1, 2, \ldots)\).
where
\[ f(x_i) = \frac{(x_i + v)_x}{x_i!} p_i^n q_i^{x_i - v} \quad (i = 1, 2) \]
are the probability mass functions of the marginal distributions \( X_1 \) and \( X_2 \) and the parameters are as defined in Section 2.1.1. Another method to obtain this probability mass function is to differentiate the given probability generating function repeatedly and then extracting the coefficient of \( x_1^{a_1} x_2^{a_2} \) [22, p. 2]:
\[ f(x_1, x_2) = \frac{1}{x_1 |x_2|} \frac{\partial^{a_1 + a_2}}{\partial z_1^{a_1} \partial z_2^{a_2}} \left\{ G(x_1, x_2)(z_1, z_2) \right\} \bigg|_{z_1, z_2 = 0}. \]
For the particular case when \( \theta_2 = 0 \), \( f(x_1, x_2) \) reduces to the following form:
\[ f(x_1, x_2) = f_1(x_1) f_2(x_2) \theta^2 \cdot F_2[1, -x_1, -x_2; 1 - x_1 - x_2, 1 - x_2; 1 - \theta_2, 1 - \theta_1], \]
where \( F_2 \) is the second-kind Appell function of two variables [43, p. 53]:
\[ F_2(a, b; c, c'; x, y) = \sum_{m, n = 0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_m(c')_n m! n!} \quad (|x| + |y| < 1) \]
and
\[ f_i(x_i) = \frac{(x_i)_x}{x_i!} p_i^n q_i^{x_i} \quad (i = 1, 2). \]
The extended bivariate negative binomial probability mass function can also be expressed in terms of Srivastava’s general triple hypergeometric series, \( F^{(3)}[x, y, z] \) [43, p. 69], by using the bivariate mixed Poisson formulation given in Section 2.3. We thus get
\[ f(x_1, x_2) = f X_1(x_1) f X_2(x_2) \]
\[ . F^{(3)} \left[ a :: b + c ; \phi \right] \left[ \begin{array}{l} a, b, c \end{array} \right] \]
\[ (a := q_1 \phi; \quad b := q_2 \phi; \quad c := q_1 q_2 (1 - \phi); \quad |\rho| < 1), \]
where
\[ \phi = \frac{\rho^2}{1 - \rho^2} \quad \text{and} \quad p_i = \frac{1}{1 + \rho_i} \quad (i = 1, 2) \]
and (see, for example, [41, p. 44, Eqs. (14) and (15)])
\[ F^{(3)}[x, y, z] = \sum_{i, j, k} A(\ell, m, n) x^\ell y^m z^n, \]
where, for convenience,
\[ A(\ell, m, n) = \prod_{i, j, k} \frac{(a_i)_{\ell} (b_j)_m (c_k)_n}{(a_i)_{\ell} (b_j)_m (c_k)_n}, \]
provided that the defining triple hypergeometric series in (10) converges. Details of the derivation of the probability mass function (9) are given in the Appendix.

### 3.2. Factorial moments and correlation

By differentiating the probability generating function (4) repeatedly,
\[ G^{(n)}(x_1, x_2) = G(x_1, x_2) \sum_{r=0}^{x_1} \sum_{s=0}^{x_2} \left( \frac{r!}{(x_1 - r)! (x_2 - s)!} \right) \left( \frac{\theta_1 + \theta_2 z_1}{1 - \theta_1 z_1 - \theta_2 z_2 - \theta_3 z_1 z_2} \right)^{x_1 - r} \left( \frac{\theta_2 + \theta_3 z_1}{1 - \theta_1 z_1 - \theta_2 z_2 - \theta_3 z_1 z_2} \right)^{x_2 - s} \]
\[ \cdot \left( \frac{\theta_3}{1 - \theta_1 z_1 - \theta_2 z_2 - \theta_3 z_1 z_2} \right) \left( x_1! x_2! \right)^{-1} \]

[Note: The image contains a table that is not transcribed into text.]
and this gives the factorial moments as

\[
\mu_{X_1,i}^{(x_1,x_2)}(1,1) = \mu_{X_1,i}^{(x_1)} \mu_{X_2,i}^{(x_2)} \sum_{i=0}^{x_1} \sum_{j=0}^{x_2} \left( \frac{(x_1)_i}{(x_1 + x_2)_i} \right) \left( \frac{(x_2)_j}{(x_1 + x_2)_j} \right) \left( \frac{x_1}{s} \right)^i \left( \frac{x_2}{j} \right)^j \left( \frac{\Theta \theta_2}{(\theta_1 + \theta_2)(\theta_2 + \theta_3)} \right)^r,
\]

where

\[
\mu_{X_i,i}^{(x)} = (x_i + v) \left( \frac{p_i}{q_i} \right)^x_i \quad (i = 1, 2)
\]

are the factorial moments of the marginal distributions of \(X_1\) and \(X_2\), respectively.

From (11) and the factorial moments of the marginals \(X_1\) and \(X_2\), the correlation coefficient is found to be

\[
\rho_{X_i,i}^{(x_1,x_2)} = \frac{v(\theta_2 + \theta_3)}{\sqrt{(x_1 + v)(x_2 + v)(1 - \theta_1)(1 - \theta_2)(\theta_1 + \theta_3)(\theta_2 + \theta_3)}} = \frac{v}{\sqrt{(x_1 + v)(x_2 + v)}} \rho_{W_1,W_2}, \tag{12}
\]

where \(\rho_{W_1,W_2}\) is the correlation coefficient of \((W_1, W_2)\).

3.3. Conditional distributions and regressions

The probability generating function of the conditional distribution of \(X_1\) (given \(X_2 = x_2\)) can be found by using

\[
G_{X_i,i}^{(x_1,x_2)}(u,v) = \frac{G_{X_i,i}^{(x_1)}(0,v)}{G_{X_i,i}^{(x_2)}(0,1)},
\]

where

\[
G_{X_i,i}^{(x_1,x_2)}(u,v) = \frac{\partial^{x_1 + x_2}}{\partial z_1^{x_1} \partial z_2^{x_2}} \left\{ G_{X_i,i}(z_1, z_2) \right\} \bigg|_{z_1 = u, z_2 = v}.
\]

Consequently, we have

\[
G_{X_i,i}^{(x_1)}(z_1) = \left( \frac{1 - p_1}{1 - p_i z} \right)^{x_1} \sum_{s=0}^{x_2} \frac{\Pr(Y_2 = x_2 - s) \Pr(W_2 = s)}{\Pr(X_2 = x_2)} \left( \frac{\theta_2 + \theta_3}{\theta_2 + \theta_3} \right)^s \left( 1 - \frac{\theta_1}{1 - \theta_1} (z - 1)^{-s} \right)^{v-s}. \tag{13}
\]

From (13), the conditional distribution of \(X_1\) given \(X_2 = x_2\) is observed to be the convolution of \(V_1\) and \(V_2\) where

\[V_1 \sim NB(x_1, p_1),\]

and \(V_2\) provides a finite mixture of convolutions:

\[(U_{1s} + U_{2s}) \quad (s = 0, 1, \ldots, x_2),\]

with

\[U_{1s} \sim \text{Bin} \left( s, \frac{\theta_3}{\theta_2 + \theta_3} \right) \quad \text{and} \quad U_{2s} \sim \text{NB}(v + s, \theta_1)\]

when \(0 < \theta_3 < 1\). In the case when \(-1 < \theta_3 < 0\), \(V_2\) is a mixture of convolutions of the pseudo-binomial and the negative binomial random variables as described in [20].

The regression of \(X_1\) on \(X_2\) is thus shown to be as follows:

\[
E[X_1|X_2 = x_2] = E[V_1] + E[V_2] = \frac{x_1 p_1}{q_1} + \sum_{s=0}^{x_2} \frac{\Pr(Y_2 = x_2 - s) \Pr(W_2 = s)}{\Pr(X_2 = x_2)} (E[U_{1s}] + E[U_{2s}])
\]

\[
= \frac{x_1 p_1}{q_1} + \frac{\theta_3 + \theta_1 \theta_2}{(1 - \theta_1)(\theta_2 + \theta_3)} \sum_{s=0}^{x_2} \Pr(Y_2 = x_2 - s) \Pr(W_2 = s) \left( \frac{1 - \frac{\theta_1}{1 - \theta_1} (z - 1)^{-s}}{1 - \theta_1} \right).\]

We note here that \(V_1\) is equivalent to \(Y_1 \sim NB(x_1, p_1)\) and the convolution of \(U_{1s}\) and \(U_{2s}\) gives the conditional distribution of \(W_1\) (given \(W_2 = s\)).

Similarly, the probability generating function of the conditional distribution of \(X_2\) (given \(X_1 = x_1\)), as well as the regression of \(X_2\) on \(X_1\), are obtained as follows:

\[
G_{X_i,i}^{(x_1)}(u,v) = \left( \frac{1 - p_2}{1 - p_i z} \right)^{x_2} \sum_{s=0}^{x_1} \frac{\Pr(Y_1 = x_1 - s) \Pr(W_1 = s)}{\Pr(X_1 = x_1)} \left( \frac{\theta_1 + \theta_3 z}{\theta_1 + \theta_3} \right)^s \left( 1 - \frac{\theta_2}{1 - \theta_2} (z - 1)^{-s} \right)^{v-s}.
\]
and
\[ E[X_2|X_1 = x_1] = \frac{x_2 p_2}{q_2} + \frac{\theta_1 + \theta_2}{1 - \theta_2} \sum_{s=0}^{x_1} \frac{\Pr(Y_1 = x_1 - s)\Pr(W_1 = s)}{\Pr(X_1 = x_1)} + \frac{\theta_1}{1 - \theta_2}. \]

Furthermore, we have
\[ E[X_2^k|X_2 = x_2] = \sum_{i=0}^{k} \sum_{s=0}^{x_2} \binom{k}{i} \frac{\Pr(Y_2 = x_2 - s)\Pr(W_2 = s)}{\Pr(X_2 = x_2)} E[V_1^k] E[W_1^r|W_2 = s]. \]

4. Canonical expansion of bivariate distributions formed by extended trivariate reduction

Many well-known bivariate generalizations of univariate Poisson, binomial, negative binomial, normal and gamma are constructed by trivariate reduction. The aforementioned univariate distributions, considered as weights for orthogonal polynomials, have generating functions for their orthogonal polynomials of the form \( G(x, t) = f(t)e^{xt}. \) Eagleson [8] has shown that their bivariate distributions obtained from trivariate reduction have canonical expansions (see Barrett and Lampard [3]; see also Lancaster [26]). Theorem 4.1 from Eagleson [8] is given below as a reference.

**Theorem** (see Eagleson [8, Theorem 4.1]). If, for a particular distribution,

(i) the orthogonal polynomials are generated by a function of the form:
\[ G(x, t) = f(t)e^{xt}, \]
where \( f(t) \) is a power series in \( t \) with \( f(0) = 1 \) and \( u(t) \) is a power series in \( t \) with \( u(0) = 0 \) and \( u'(0) = 1, \)

(ii) the distribution is additive, and

(iii) a bivariate distribution is generated by using the additive property (1),

then the matrix of correlations of the pairs of orthonormal polynomials on the marginals is diagonal. Further, \( \rho_{rs} = \rho_{r} \) depends only on the normalizing factor of the \( r \)th orthogonal polynomial.

The following result extends Eagleson’s result (the above Theorem) to bivariate distributions which can be formed by the extended trivariate reduction given in Section 2.1.

**Result 1.** If \( (W_1, W_2) \) has a bivariate distribution with canonical expansion in terms of orthogonal polynomials, then another bivariate distribution \( (X_1, X_2) \) generated by using the additive property:
\[ X_1 = Y_1 + W_1 \quad \text{and} \quad X_2 = Y_2 + W_2, \]
where \( Y_1 \) and \( Y_2 \) are independent, also has a canonical expansion in terms of orthogonal polynomials.

**Proof.** Let \( \xi_r(u) \) denote the \( r \)th orthonormal polynomial of the \( r \)th orthogonal polynomial \( \xi_s(u) \) on a distribution \( U \) and
\[ c_r^{(U)}(u) = \int_{-\infty}^{\infty} \xi_r(u)dF(u), \]
where \( F(u) \) is the distribution function of \( U \). We also let \( \rho_{rs}^{(U,V)} \) denote the correlation of a \((r, s)\) pair of such orthonormal polynomials on the marginals of the bivariate distribution of \((U, V)\), that is,
\[ \rho_{rs}^{(U,V)} = E[\xi_r(u)\xi_s(u)]. \]

Extending the above Theorem of Eagleson (see [8, p. 1211]), we obtain
\[ \left[ c_1^{(X)} c_2^{(X)} \right]^2 \rho_{rs}^{(X, X)} = \int \xi_1(x_1)\xi_2(x_2)dF(x_1, x_2) \]
\[ = \int \int \left( \sum_{r=0}^{\infty} \binom{r}{i} \xi_i(y_1)\xi_{r-i}(w_1) \right) \left( \sum_{s=0}^{\infty} \binom{s}{j} \xi_j(y_2)\xi_{s-j}(w_2) \right) \cdot dF_1(y_1)dF_2(y_2)dF_{12}(w_1, w_2) \]
\[ = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \binom{r}{i} \binom{s}{j} \int \xi_i(y_1)\xi_j(y_2)dF_1(y_1)dF_2(y_2) \cdot \left( \int \xi_{r-i}(w_1)\xi_{s-j}(w_2)dF_{12}(w_1, w_2) \right). \]

Using a corollary from Lancaster’s work [27, p. 535], which states that a necessary and sufficient condition for independence of the marginal variables of a bivariate statistical distribution is that
\[ \rho_{ij} = 0 \quad (i > 0; \ j > 0), \]
function is found to be given by
\[ \frac{[c(X^1)_t,c(X^1)_j]^\frac{1}{2}}{r_{ij}^{s}} = \int \frac{\zeta_i(w_1)\zeta_j(w_2)dF(w_1,w_2)}{\delta_{ij}[c(w_1)c(w_2)]^\frac{1}{2}}, \]
where \( \delta_{ij} \) is the Kronecker’s delta. This clearly indicates that the matrix of correlations is diagonal. \( \square \)

**Remark 1.** In general, the existence of the canonical expansion of a bivariate distribution may be proved by using the criterion of Brown [4] which requires that the conditional moments \( E[X^i|Y] \) and \( E[Y^i|X] \) must be polynomials with degree less than or equal to \( n \). This criterion may not be easy to apply.

**Result 1** shows that the extended bivariate negative binomial distribution has a canonical expansion in terms of orthogonal polynomials. We next derive this canonical expansion.

The factorial moment generating function for
\[ f(x) \frac{m_r(x;v,p)}{(v)_r} \]
is given by
\[ \sum_{x=0}^{\infty} (1+t)^rf(x) \frac{m_r(x;v,p)}{(v)_r} = (-p)^{-r} \left( \frac{tp}{q} \right)^r \left( 1 - \frac{tp}{q} \right)^{-(v+r)}, \]
where
\[ m_r(x; v, p) = (v)_r \cdot {}_2F_1 \left( -r; -x; v; 1 - \frac{1}{p} \right) \]
is the \( r \)th Meixner polynomial and
\[ f(x) = \frac{(v)_x}{x!} p^x q^r \]
is the negative binomial probability mass function. The factorial moment generating function of the extended bivariate negative binomial distribution from (4) is given by
\[ H(t_1, t_2) = (1 - A_1 t_1)^{-x_1 - r} (1 - A_2 t_2)^{-x_2 - r} \left( 1 - \frac{(A_3 + A_1 A_2) t_1 t_2}{(1 - A_1 t_1)(1 - A_2 t_2)} \right)^{-v}, \]
where
\[ A_1 = \frac{p_1}{q_1}, \quad A_2 = \frac{p_2}{q_2} \quad \text{and} \quad A_3 = \frac{\theta_3}{\theta}. \]

Following Kocherlakota and Kocherlakota [22, p. 135], \( H(t_1, t_2) \) is expanded to give the following result:
\[ H(t_1, t_2) = \sum_{i=0}^{\infty} \frac{(v)_i}{i!} \delta^i (A_1 t_1)^i (1 - A_1 t_1)^{-x_1 - r - i} (A_2 t_2)^i (1 - A_2 t_2)^{-x_2 - r - i}, \]
where
\[ \delta = \frac{\rho(w_1,w_2) \sqrt{(1 - \theta_1)(1 - \theta_2)}}{\theta \sqrt{A_1 A_2}}. \]

Using the relation in (14), the canonical expansion for the extended bivariate negative binomial distribution probability mass function is found to be given by
\[ f(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2) \sum_{i=0}^{\infty} \frac{(v)_i}{i!} \frac{\rho_i^{s}(w_1,w_2)}{(x_1 + v_i)(x_2 + v_i)} m^i_1(x_1; x_1 + v, p_1) m^i_2(x_2; x_2 + v, p_2), \]
\[ = f_{X_1}(x_1)f_{X_2}(x_2) \sum_{i=0}^{\infty} \frac{(v)_i}{\sqrt{(x_1 + v_i)(x_2 + v_i)}} \rho_i^{s}(w_1,w_2) m^i_1(x_1; x_1 + v, p_1) m^i_2(x_2; x_2 + v, p_2). \]
where
\[ f_{X}(x_{j}) = \frac{(x_{j} + v)_{\nu_{j}}}{x_{j}!} p_{j}^{x_{j}} q_{j}^{x_{j}+r} \quad (j = 1, 2) \]
are the marginal probability mass functions of \( X_{1} \) and \( X_{2} \), and
\[ m_{i}^{j}(x_{j}; x_{j} + v, p_{j}) = \frac{m_{i}(x_{j}; x_{j} + v, p_{j})}{\sqrt{(x_{j} + v)! p_{j}^{-1}}} \quad (j = 1, 2) \]
is the \( i \)th orthonormal Meixner polynomial.

5. Quadrant dependence

Various forms of bivariate dependence have been used for statistical analysis. One of them is positive quadrant dependence, introduced by Lehmann [29], which is a very useful measure of dependence. Compared to other types of dependence, positive quadrant dependence is easier to show as seen from its definition given as follows.

**Definition.** Two random variables \( X \) and \( Y \) are said to be positive quadrant dependent if
\[
\Pr(X \leq x, Y \leq y) \geq \Pr(X \leq x)\Pr(Y \leq y) \quad (\forall x; \forall y). \tag{17}
\]
Jensen [16] has extended (17) to regions other than quadrant with the concept of positive dependence when the marginal distributions are identical. Jensen’s definition of positive dependence is recalled here as follows.

Two random variables \( X \) and \( Y \) are said to be positively dependent if
\[
\Pr(X \in A, Y \in A) \geq \Pr(X \in A)\Pr(Y \in A)
\]
for every measurable set \( A \) with respect to the marginal measure.

It is easy to show that the extended bivariate negative binomial distribution is positively dependent from the canonical expansion of its joint probability mass function given by (16) when marginals are identical. We now show that the extended bivariate negative binomial distribution is positive quadrant dependent when the marginal parameters are different.

**Result 2.** The extended bivariate negative binomial distribution with the joint probability generating function (8) is positive quadrant dependent when
\[
0 < \rho_{(w_{1}, w_{2})} < 1.
\]

**Proof.** First of all, we note that
\[
G_{X,Y}(z_{1}, z_{2}) \geq G_{X}(z_{1})G_{Y}(z_{2})
\]
implies that
\[
\Pr(X \leq x, Y \leq y) \geq \Pr(X \leq x)\Pr(Y \leq y) \quad (\forall x; \forall y),
\]
that is, positive quadrant dependence. This follows by extracting the \((x, y)\)th term from the probability generating function. Upon rewriting (8) by using the relation in (15), we obtain
\[
G(z_{1}, z_{2}) = H(z_{1} - 1, z_{2} - 1),
\]
so that
\[
G_{X,Y}(z_{1}, z_{2}) = [1 + A_{1}(1 - z_{1})]^{-z_{1}} - [1 + A_{2}(1 - z_{2})]^{-z_{2}} \times \\
\left[ 1 + \sum_{i=1}^{\infty} \frac{(v_{i})}{i!} \left( \rho_{(w_{1}, w_{2})} \sqrt{(1 - \theta_{1})(1 - \theta_{2})} \right)^{i} \frac{A_{1}A_{2}(1 - z_{1})(1 - z_{2})}{\Theta A_{1}A_{2}} \right],
\]
where
\[
A_{1} = \frac{p_{1}}{q_{1}} \quad \text{and} \quad A_{2} = \frac{p_{2}}{q_{2}}.
\]
Since
\[
|z_{i}| \leq 1 \quad (i = 1, 2),
\]
we have
\[
1 - z_{i} \geq 0
\]
and we thus find that the terms $A_i(1 - z_i)$ are also positive. When
\[ 0 < \rho_{(w_1, w_2)} < 1, \]
the infinite series in braces in (18) is positive. Hence (18) implies that
\[ G_{(X_1, X_2)}(z_1, z_2) \geq G_{X_1}(z_1)G_{X_2}(z_2). \]
From the remark at the beginning of the proof, we conclude that the extended bivariate negative binomial is positive quadrant dependent when
\[ 0 < \rho_{(w_1, w_2)} < 1. \]

**Remark 2.** Positive quadrant dependence of the bivariate negative binomial distributions of Edwards and Gurland [9] and Subrahmaniam [48] follow upon setting
\[ \alpha_1 = \alpha_2 = 0. \]

**Remark 3.** For the extended trivariate reduction (2), it is easy to show that, if $(W_1, W_2)$ is positive quadrant dependent, then $(X_1, X_2)$ is also positive quadrant dependent.

### 6. Computer generation of bivariate samples

In this section, we give the algorithms to generate random samples from extended bivariate negative binomial distribution. The algorithm is also applicable to generate random samples for the extended bivariate binomial and gamma distributions.

#### 6.1. Mixture method

By the following formulation:
\[
X_1 = Y_1 + W_1 \quad \text{and} \quad X_2 = Y_2 + W_2,
\]
the general form of probability generating function for the extended bivariate negative binomial distribution is given as in (3). The correlation for the extended bivariate negative binomial distribution is given by
\[
\rho_{(X_1, X_2)} = \frac{\sqrt{a_1 + \rho} \sqrt{a_2 + \rho}}{\rho_{(W_1, W_2)}},
\]
where $\alpha_1, \alpha_2$ and $\rho$ are the corresponding index parameters for $Y_1, Y_2$ and $(W_1, W_2)$ distributions and $\rho_{(W_1, W_2)}$ is the correlation of the bivariate distribution of $(W_1, W_2)$. Extended bivariate binomial and gamma distributions can also be shown to have the same form of correlation relation.

For any of the bivariate distributions, given the marginals:
\[
X_1 \sim g(x, \theta_1) \quad \text{and} \quad X_2 \sim g(\beta, \theta_2)
\]
as well as the correlation $\rho_{(X_1, X_2)}$, it can be deduced that
\[
Y_1 \sim g(x - v, \theta_1) \quad \text{and} \quad Y_2 \sim g(\beta - v, \theta_2),
\]
\[
W_1 \sim g(v, \theta_1) \quad \text{and} \quad W_2 \sim g(v, \theta_2)
\]
and
\[
\rho_{(W_1, W_2)} = \frac{\sqrt{(a_1 + \rho)(a_2 + \rho)}}{\sqrt{\rho_{(X_1, X_2)}}},
\]
with $g(\cdot)$ being one of the corresponding univariate distributions. Ong (\cite{34,35}) has given several mixture models as well as algorithms for computer generation of the bivariate negative binomial, binomial and gamma distributions of $(W_1, W_2)$ with given marginals and correlation. Utilizing this and the above formulation, an algorithm to generate bivariate data from one of the three distributions with different marginals is given below.

**Algorithm 1**

1. Set
\[
0 < v < \min(a, \beta), \quad \alpha_1 = a - v, \quad \alpha_2 = \beta - v
\]
and
\[ \rho_{(w_1, w_2)} = \frac{\sqrt{(x_1 + v)(x_2 + v)}}{v}\rho_{(x_1, x_2)}. \]

2. Generate \( y_1 \sim g(x_1, \theta_1) \) and \( y_2 \sim g(x_2, \theta_2) \).
3. Use known marginals:
   \[ W_1 \sim g(v, \theta_1), \quad W_2 \sim g(v, \theta_2) \quad \text{and} \quad \rho_{(w_1, w_2)} \]
   in the bivariate negative binomial algorithm from Ong and Lee [37] to generate \((w_1, w_2)\).
4. \( x_1 = y_1 + w_1 \) and \( x_2 = y_2 + w_2 \).

6.2. Conditional distribution technique

The conditional distribution of \( X_1 \) (given \( X_2 = x_2 \)) is the convolution of \( V_1 \) and \( V_2 \) as given in Section 3.3. We are given the marginals:
\[ X_1 \sim NB(\alpha, p_1), \quad X_2 \sim NB(\beta, p_2) \quad \text{and} \quad \rho_{(x_1, x_2)}. \]
When \( 0 < \theta_3 < 1 \), it is found that
\[ V_1 \sim NB(\alpha - v, p_1), \]
\[ U_{1s} \sim Bin\left(s, \frac{\theta_3}{\theta_2 + \theta_3}\right) \quad \text{and} \quad U_{2s} \sim NB(v + s, \theta_1) \quad (s = 0, 1, \ldots, x_2) \]
and
\[ Y_2 \sim NB(\beta - v, p_2) \quad \text{and} \quad W_2 \sim NB(\gamma, p_2). \]
When \( -1 < \theta_3 < 0 \), \( V_2 \) is a mixture of convolutions between a pseudo-binomial and a negative binomial random variables which can be easily generated using the standard inverse transform method.

Algorithm 2

1. Set \( 0 < v < \min(\alpha, \beta) \), \( x_1 = \alpha - v \), \( x_2 = \beta - v \).
2. Set \( \theta_1, \theta_2 \) and \( \theta_3 \) such that
   \[ \rho_{(x_1, x_2)} = \frac{v(\theta_3 + \theta_1 \theta_2)}{\sqrt{(x_1 + v)(x_2 + v)(1 - \theta_1)(1 - \theta_2)(\theta_1 + \theta_3)(\theta_2 + \theta_3)}}. \]
3. Generate \( x_2 \sim NB(\beta, p_2) \) and \( v_1 \sim NB(\alpha, p_1) \).
4. Set \( v_2 = 0 \). For \( s = 0 \) to \( x_2 \),
   (a) When \( 0 < \theta_3 < 1 \),
   (i) Generate \( u_{1s} \sim B\left(s, \frac{\theta_3}{\theta_2 + \theta_3}\right) \) and \( u_{2s} \sim NB(v + s, \theta_1) \).
   (ii) \( v_2 = v_2 + \left(\frac{Pr(Y_2 = x_2 - s)Pr(W_2 = s)}{Pr(X_2 = x_2)}\right)(u_{1s} + u_{2s}) \).
   (b) When \( -1 < \theta_3 < 0 \), generate \( v_2 \) using inverse transform method based on the probabilities given in [20].
5. \( x_1 = v_1 + v_2 \).

7. Applications

In this section, we illustrate the applications of the two formulations of the extended bivariate negative binomial distribution for accident data bearing in mind that it is also applicable in other contexts.

7.1. Extended trivariate reduction

Suppose that accidents or injuries are due to (a) individual characteristics and (b) environmental factors [2]. Let \( Y_1 \) and \( Y_2 \) represent the number of accidents due to cause (a) in two different time periods. Suppose that accidents due to cause (b) vary from one time period to another as a pair of correlated random variables \((W_1, W_2)\). The total number of accidents in each period will be given by
\[ X_1 = Y_1 + W_1 \quad \text{and} \quad X_2 = Y_2 + W_2. \]
If \( Y_1 \) and \( Y_2 \) are assumed to be Poisson-distributed but due to individual characteristics, accident proneness varies from individual to individual as a gamma distribution, then \( Y_1 \) and \( Y_2 \) will have the negative binomial distributions. A similar
reasoning for accidents due to environmental factors lead to the assumption that \((W_1, W_2)\) has the bivariate negative binomial distribution of Edward and Gurland [9]. Hence, accidents in the two time periods will have the extended bivariate negative binomial distribution.

7.2. Mixed Poisson formulation

Let \(X\) and \(Y\) represent the number of accidents or injuries sustained by a group of individuals in two different time periods, each of unit length, with Poisson distributions \(\text{Poi}(\lambda_1)\) and \(\text{Poi}(\lambda_2)\), respectively. Suppose that the population consists of individuals where the proneness of each individual to accidents varies from individual to individual (see Edwards and Gurland [9] and Subrahmaniam [48]), that is, \(\lambda_i, i = 1, 2\) differs from individual to individual. If \(\lambda_1\) and \(\lambda_2\) have a joint bivariate gamma distribution given by (6) we get the extended bivariate negative binomial distribution.

7.3. Numerical examples

Two examples of the extended bivariate negative binomial distribution fits to a simulated data set and the rain-forest data set (see Holgate [15]) are considered in this section. The parameters are estimated by the maximum likelihood estimation (MLE) and the fits are compared with Edwards and Gurland’s bivariate negative binomial distribution. The log likelihood function is maximized using the numerical method of simulated annealing to obtain globally optimum parameter estimates. Suitable bounds are set for the unbounded parameters \(a_1, a_2\) and \(\theta_3\) to assist in the numerical parameter searches. Bounds for the parameters \(p_1, p_2\) and \(\theta_3\) are as given in Section 2.3.

**Example 1.** A sample of size 500 is simulated from the extended bivariate negative binomial distribution with \(p_1 = 0.4, p_2 = 0.5, \theta_3 = 0.3, x_1 = 0.5, x_2 = 2.5\) and \(v = 1.0\), where the marginals \(X_1 \sim NB(0.4, 1.5)\) and \(X_2 \sim NB(0.5, 3.5)\) clearly have different index parameters. Simulation is done according to the algorithm outlined in Section 6.1. Observed frequencies for the data are shown in Table 1.

The extended bivariate negative binomial and Edwards and Gurland’s bivariate negative binomial distributions are fitted to the data with grouping of frequencies at the cell (16, 8). The comparison of the fittings is made based on the chi-square, \(\chi^2\) goodness-of-fit statistic. The parameter estimates and corresponding \(\chi^2\) values as well as degrees of freedom (d.f.) are given in Table 2. The expected frequencies for these two distributions are then given in Table 3.

It is obvious from the \(\chi^2\) values in Table 2 that bivariate negative binomial could not give a satisfactory fit (\(p\)-value = 0.09) when the index parameters for the marginals are different as compared to the extended bivariate negative binomial (\(p\)-value = 0.56).

**Example 2.** The abundance of two different plant species in the rain-forest data [15] can be due to individual growth factor and environmental factors such as climate and space. We fit the data to the extended bivariate negative binomial and Edwards and Gurland’s bivariate negative binomial distributions. The ML estimates of the parameter are tabulated here as follows.

### Table 1
Simulated sample of size 500 from extended bivariate negative binomial distribution. \((p_1 = 0.4, p_2 = 0.5, \theta_3 = 0.3, x_1 = 0.5, x_2 = 2.5, v = 1.0)\)

<table>
<thead>
<tr>
<th>(x_2)</th>
<th>0</th>
<th>1</th>
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**Note:** The dashed lines indicate grouping of the data for the \(\chi^2\) goodness-of-fit test to yield a minimum expected frequency of 1.
Table 2
Extended bivariate negative binomial and bivariate negative binomial: parameter estimates and $\chi^2$ values.

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EBNB: extended bivariate negative binomial.
BNB: bivariate negative binomial.

Table 3
Expected extended bivariate negative binomial (bivariate negative binomial) frequencies: simulated data.

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Observed and expected frequencies for rain-forest data.

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<td>$x_i \geq 5$</td>
<td>$x_i \leq 4$</td>
<td>$x_i \geq 5$</td>
<td>$x_i \geq 5$</td>
</tr>
<tr>
<td>0</td>
<td>1.18</td>
<td>0.78</td>
<td>0.46</td>
<td>0.39</td>
<td>0.26</td>
</tr>
<tr>
<td>1</td>
<td>1.39</td>
<td>0.81</td>
<td>0.51</td>
<td>0.40</td>
<td>0.27</td>
</tr>
<tr>
<td>2</td>
<td>1.39</td>
<td>0.78</td>
<td>0.51</td>
<td>0.40</td>
<td>0.27</td>
</tr>
<tr>
<td>3</td>
<td>1.39</td>
<td>0.78</td>
<td>0.51</td>
<td>0.40</td>
<td>0.27</td>
</tr>
<tr>
<td>4</td>
<td>1.39</td>
<td>0.78</td>
<td>0.51</td>
<td>0.40</td>
<td>0.27</td>
</tr>
</tbody>
</table>

EBNB: bivariate non-central negative binomial.

(i) Extended bivariate negative binomial distribution:

\[ p_1 = 0.308341, \quad p_2 = 0.245690, \quad \theta_3 = 0.225283, \]

\[ x_1 = 1.463361, \quad x_2 = 1.300684, \quad \nu = 0.638345, \]

with marginals $X_1 \sim NB(0.3038,2.1017)$ and $X_2 \sim NB(0.2457,1.9390)$.

(ii) Edwards and Gurland’s bivariate negative binomial

\[ p_1 = 0.288067, \quad p_2 = 0.208444, \quad \theta_3 = 0.003166, \quad \nu = 2.336087, \]

with marginals $X_1 \sim NB(0.2881,2.3361)$ and $X_2 \sim NB(0.2084,2.3361)$.

The expected frequencies obtained from both distributions are shown in Table 4. Expected frequencies from the Type II bivariate non-central negative binomial distribution fit from Ong and Lee [38] are also given for comparison.

We note that for the rain-forest data the marginal distributions for the extended bivariate negative binomial and bivariate negative binomial are similar. As expected, the fit by extended bivariate negative binomial yields a smaller $\chi^2$ value compared to bivariate negative binomial since more flexibility is allowed for the marginals. This $\chi^2$ value is also smaller than the $\chi^2$ value obtained from the Type II bivariate non-central negative binomial distribution.

8. Concluding remarks and observations

This paper has considered an extension of trivariate reduction and a mixed Poisson formulation to obtain a bivariate negative binomial distribution with different marginal parameters and the correlation coefficient in the range $(0,1)$. Distributional properties have been derived. For the extended trivariate reduction, a result on the existence of canonical expansion of the bivariate distribution has been obtained. The extended bivariate negative binomial distribution is shown to be positive quadrant dependent. The generation of bivariate data with varying dependence has also been considered. A fit with simulated data shows that when the negative binomial marginals are very different, the extended bivariate negative binomial distribution is to be preferred. For the rain-forest data, the extended bivariate negative binomial distribution fits better than the bivariate negative binomial distribution as judged by the $\chi^2$ value (see also a recent work by Nadarajah et al. [33]).

We note that a bivariate binomial distribution can be constructed in a similar manner as the extended bivariate negative binomial distribution. Furthermore, bivariate continuous distributions such as the bivariate normal and gamma distributions...
can also be constructed through this extended trivariate reduction method by utilising the moment generating functions instead of probability generating functions.

Here, in our present investigation, we have successfully applied such special functions as the Laguerre and Meixner polynomials (as well as hypergeometric functions in one, two and three variables) to provide a systematic study of an extension of trivariate reduction and a mixed Poisson formulation for obtaining a bivariate negative binomial distribution with different marginal parameters and the correlation coefficient in the range (0,1). Traditionally, various classes of special functions and polynomials of mathematical physics and analytic number theory in one and more variables have found interesting applications not only in mathematical, physical and engineering sciences, but surely also in probability theory and other disciplines in the statistical sciences (see, for example, [40–42]; see also [18,19]). We choose to conclude this paper by referring the interested reader to several recent investigations on the subject by (among others) Gupta et al. ([12,13]) (see also [10,30]), Lee et al. [28] and Srivastava et al. ([44–46]).

Appendix A

A.1. Canonical expansion

The canonical expansion of a bivariate distribution is defined as follows.

Let $h(x,y)$ be a bivariate probability density function with marginal probability density functions $f(x)$ and $g(y)$ where the parameters have been suppressed for simplicity. Let $\{\psi_i^{(1)}(x)\}$ and $\{\psi_i^{(2)}(y)\}$ be complete set of orthonormal functions with respect to $f(x)$ and $g(y)$ respectively. Then $h(x,y)$ can be expanded as a double series in the form:

$$h(x,y) = f(x)g(y) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \rho_{ij} \psi_i^{(1)}(x) \psi_j^{(2)}(y),$$  \hspace{1cm} (A.1)

where

$$\rho_{ij} = \int \int h(x,y) \psi_i^{(1)}(x) \psi_j^{(2)}(y) \, dx \, dy$$

and the series:

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \rho_{ij}^2$$

is convergent [26]. Let

$$\phi^2 = \int \left( \frac{dH(x,y)}{df(x)dg(y)} \right)^2 \, df(x) \, dg(y) - 1,$$

where $H(x,y), f(x)$ and $G(y)$ are the distribution functions corresponding, respectively, to $h(x,y), f(x)$ and $g(y)$. If $\phi^2$ is bounded (see Lancaster [26]), then (A.1) can be expressed in the canonical form such that

$$\rho_{ij} = 0 \quad (i \neq j),$$

that is, the coefficient matrix $[\rho_{ij}]$ is diagonal. The double series in (A.1) yields

$$h(x,y) = f(x)g(y) \sum_{i=0}^{\infty} \rho_i \psi_i^{(1)}(x) \psi_i^{(2)}(y) \quad (\rho_i := \rho_{ii}),$$  \hspace{1cm} (A.2)

with

$$\phi^2 = \sum_{i=1}^{\infty} \rho_i^2.$$  

The series in (A.2) is the canonical expansion of the bivariate probability density function $h(x,y)$ and $\rho_i$ is known as the $i$th canonical coefficient.

A.2. Probability mass function of the extended bivariate negative binomial distribution

In terms of the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ of order (or index) $\alpha$ and degree $n$ in $x$, defined by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{n + \alpha}{n - k} \frac{(-x)^k}{k!} = \frac{(n + \alpha)}{n} \frac{\Gamma(-n; \alpha + 1; x)},$$

where

$$\left\{\begin{array}{c}
1F1(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} = \lim_{\beta \to \infty} \left\{\beta F1 \left(a, b; c; \frac{z}{\beta} \right) \right\}
\end{array}\right.$$
denotes the Kummer confluent hypergeometric function, the canonical form of the bivariate gamma distribution probability density function [25, p. 41] with the moment generating function (6) is given by

\[ f(x, y) = \frac{e^{-x_1 y_1^{-1}} e^{-x_2 y_2^{-1}}}{I(\alpha_1 + v) I(\alpha_2 + v)} \sum_{k=0}^{\infty} a_k \rho^k L_k^{(\alpha_1 + \gamma)}(x)L_k^{(\alpha_2 + \gamma)}(y), \]

where

\[ a_k = \frac{k!(\gamma)_k}{(\alpha_1 + v)_k (\alpha_2 + v)_k}. \]

By applying the following result of Srivastava [39, p. 306, Eq. (2.3)] (see also Srivastava and Manocha [43, p. 221, Eq. (18)])

\[ \sum_{n=0}^{\infty} \frac{n!(\lambda)_n}{(\alpha + 1)_n (\beta + 1)_n} L_n^{(\lambda)}(x) L_n^{(\beta)}(y) z^n = (1 - z)^{-\lambda} \exp \left( -\frac{xz}{1-z} \right) \cdot \Phi(3) \left[ \begin{array}{c} x - \lambda + 1, \beta; \alpha + 1, \beta + 1; \frac{xz}{1-z}, \frac{yz}{1-z} \end{array} \right] (|z| < 1), \]

to the infinite probability sum in the bivariate density function and setting

\[ x = \beta_1 \lambda \quad \text{and} \quad y = \beta_2 \theta, \]

we obtain

\[ f(\lambda, \theta) = \frac{e^{-\beta_1^2 \lambda^2} \lambda^2_{2+1} \lambda_{2+1}^{-1}}{I(\alpha_1 + v) I(\alpha_2 + v)} \cdot (1 - \rho^2)^{-\lambda} \exp \left( -\frac{\beta_1 \lambda^2}{1 - \rho^2} \right) \cdot \Phi(3) \left[ \begin{array}{c} \beta_1, \beta_2, \gamma; \alpha; \beta; 1; \frac{\beta_1 \lambda^2}{1 - \rho^2}, \frac{\beta_2 \lambda^2}{1 - \rho^2}, \frac{\beta_1 \beta_2 \lambda^2}{1 - \rho^2} \end{array} \right], \]

where \( \Phi(3) \) is a certain confluent hypergeometric function of three variables defined by Srivastava [39] by (see, for example, Srivastava and Manocha [43, p. 222])

\[ \Phi(3)[x, \beta; \gamma, \delta; x, y, z] = \sum_{l,m,n=0}^{\infty} \frac{(x)_l (\beta)_{m+n} x^l y^m z^n}{l! m! n!}, \]

which can, in fact, be expressed in terms of a very specialized case of Srivastava's general triple hypergeometric series \( F(3)[x, y, z] \) defined by (10).

By the mixed Poisson formulation, the joint probability mass function of the extended bivariate negative binomial distribution is then given by

\[ f(x_1, x_2) = \int_0^\infty \int_0^\infty e^{-\lambda x_1} x_1! x_2! f(\lambda, \theta) d\lambda d\theta = \sum_{l,m,n=0}^{\infty} \frac{(x_1)_l (\beta)_{m+n} x_1^l y^m z^n}{l! m! n!} \int_0^\infty \frac{\beta_1^2 \lambda^2 + \beta_2^2 \gamma^2}{x_2^l - e^{-\beta_2 \theta} \theta^{2+1} \gamma^{m+n}} d\theta, \]

which gives the probability mass function (9) in terms of one of Srivastava's general triple hypergeometric series as defined by (10).

References
